A progressive domain expansion method for solving optimal control problem

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ABSTRACT

Electricity generation at the hydropower stations in Nigeria has been below the expected value. While the hydro stations have a capacity to generate up to 2,380 MW, the daily average energy generated in 2017 was estimated at around 846 MW. A factor responsible for this is the lack of a proper control system to manage the transfer of resources between the cascaded Kainji-Jebba Hydropower stations operating in tandem. This paper addressed the optimal regulation of the operating head of the Jebba hydropower reservoir in the presence of system constraints, flow requirement and environmental factors that are weather-related. The resulting two-point boundary value problem was solved using the progressive expansion of domain technique as against the shooting or multiple shooting techniques. The results provide the optimal inflow required to keep the operating head of the Jebba reservoir at a nominal level, hence ensuring that the maximum number of turbo-alternator units are operated.

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1. INTRODUCTION

The Nigerian national power grid is powered from 29 generation stations having an installed capacity of 12,910.40MW but a relatively lower generation capacity of roughly 7,652.6MW. 19% of the installed capacity is provided by the more reliable and cheap hydropower plants. The three major hydropower plants in Nigeria are the Kainji hydroelectric power station (KHEPS), the Jebba hydroelectric power station (JHEPS) and the Shiroro hydroelectric power station [1-3].

The KHEPS located on 09°51′45″N, 04°36′48″E and JHEPS on 9°08′08″N, 4°47′16″E are in tandem and managed by the mainstream energy solution. They are built on the River Niger and operated in cascade. Unfortunately, the generation efficiency of JHEPS has been lower than expected, the daily average energy generated in 2017 was estimated at around 846 MW [4]. The two power stations operate in cascade but lack a control system regulating their operation. They are being managed by the experience and intuition of the operators [2, 5]. From the operational report, there are occasions where some units at JHEPS are shut down if the release from KHEPS is low [6].

Based on the available information gathered, the problem of real-time optimal management of resources between KHEPS and JHEPS has not been solved. Despite all the research efforts made so far, operators still rely on intuitive water release rules to maximize power generation regularly. It is obvious that the operational procedure based on intuition and experience can leads to poor utilization of available energy resources [5, 7]. There are no proper scientifically motivated techniques for the management of resources between the two stations. Most models and optimization techniques earlier proposed are parameter optimization technique which is not sufficient for managing a dynamic system nonlinear system [8-11].

A difficult problem encountered by the plant operators occurs in the management of water flows between the two reservoirs. Experience from years of operations seems to support this approach given the fact that a necessary and enough condition for the proper operation of the turbo-alternators is that the operating net head must satisfy the requirements for acceptable energy conversion by the turbine. Therefore, the optimal control problem solved in this work is the determination of the best control vector and resulting state trajectories which minimizes certain performance index, subject to system constraints. In this case, the optimal control problem results, whereby a control signal is desired that will force the reservoir head at JHEPS to move from an initial point to the desired point in a finite time and subject to constraints imposed by the system dynamics.

Unfortunately, many problems that are rooted in nonlinear optimal control theory do not have computable solutions or they have solutions that may be obtained only with a great deal of computing effort [12, 13]. The solution via analytical means also seems not feasible except by numerical means. Numerical solutions to optimal control problems are either through direct or indirect methods. In the direct methods, the infinitely dimensional state and controls are discretized. The indirect method applies calculus of variation to set up necessary conditions that must be satisfied by the optimal control. Calculus of variation, together with Pontryagin's minimum principles are used to setup optimality conditions. These conditions produce optimal control canonical equations such that their solution ensures that an optimum point has been reached [14, 15].

The indirect approach leads to a nonlinear two-point boundary-value problem. The control task then reduces to the solution of a boundary value problem. There are different approaches with associated advantages and disadvantages. In all the solution techniques, an initial guess is used to obtain a solution in which one or more of the necessary optimality conditions are not satisfied. The solution is then used to adjust the initial guess to make the next solution come closer to satisfying all the necessary conditions. If the steps are repeated and the iterative procedure converges, the necessary conditions will eventually be reached [16-18].

Many authors had proposed methods of solving an optimal control problem, these methods can be in the form of nonlinear programming, shooting method, forward backward sweep, steepest descent, conjugate gradient, dynamic programming, the variation of externals, quasi-linearization, gradient projection, collocation, etc [19, 20]. There has been no perfect method as each has its own advantages and disadvantages. For example, the forward-backward sweep (FBS) works only if the Lipschitz constants for the state, costate and control variables are small enough or the time interval is very small. Likewise, the covergence of the shooting method depends on the numerical procedure and the initial data set, else there will be no solution [21].

The multiple shooting and parallel shooting techniques were earlier explored in [22-27] by resetting the problem and increasing the single initial value problem to a family of initial value problems configured so as to limit the effect of the growth of computational errors. The outcome resulted in a method that increased the number of guesses which were much fewer than the methods that depended on variational and other approximation methods that for the same accuracy may involve the solution of large linear or nonlinear equations that have dimensionality several orders of magnitude when compared with the corresponding shooting method. The method was applied successfully in the modeling of distributed parameter systems and proved to be very efficient, accurate and fast. The progressive domain expansion method (PDEM) proposed in this paper is another modification shooting method. It is less computational and feasible in solving the optimal control problem at JHEPS.

2. RESEARCH METHOD

The solution to an optimal control problem requires a proper model of the system dynamics in the form of a differential equation or difference equation. A suitable performance index with its associated constraints must be developed; there should also be a numerical technique for solving the model equation and a procedure for solving the resulting boundary value problem. The system can be described by the block diagram of Figure 1 and the dynamical model of (1). This model was earlier presented in [5]. The block diagram presents the reservoir dynamics and power generation as a function of the operating head. As presented in [5], the electrical power form JHEPS is represented in (1) and (2), showing that it is a function of the operating head.

$$P(t) = \sqrt{2}\,\eta\rho A_2\, g^{3/2} h^{3/2}(t) \quad (1)$$

$$P(t) = \mathrm{K}(h(t)) \quad (2)$$

where K represents a scalar valued function.

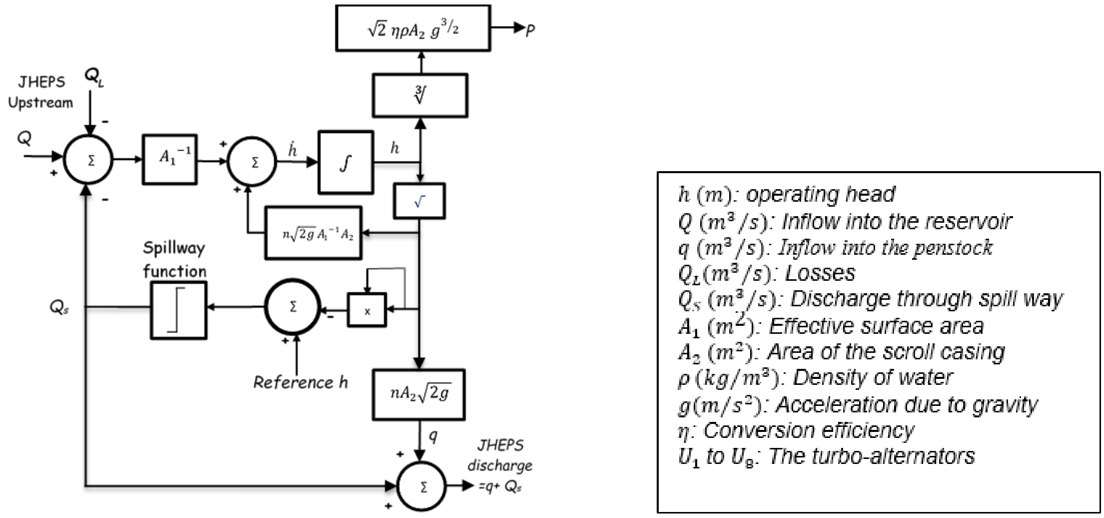


Figure 1. Block diagram of the JHEPS reservoir dynamics

Therefore, (3) presents the differential equation (model equation) that describes the operating head dynamics,

$$\frac{dh(t)}{dt} = -\,nA_1^{-1}A_2\sqrt{2gh(t)} + A_1^{-1}(Q(t) - Q_L(t) - Q_s(t)) \quad (3)$$

$$\dot{h}(t) = f(h(t), u(t)) \quad (4)$$

where f also represents a scalar valued function.

The optimal control of the JHEPS devolves into the determination of best control vector $u(t)\,\epsilon\,U(t)$ from KHEPS, which compels the dynamical system $\dot{h}(t) = f\big(h(t), u(t),\, t\big)$ at JHEPS to follow an optimal trajectories $h^*(t)$ that minimize specified performance indices. The system is nonlinear and exists in the continuous-time domain. The optimal control problem is the Lagrange problem. As earlier mentioned, while solving the most optimal control problems, the solution can be intractable by analytical methods; the engineers depend on numerical procedures. The numerical solution used in determining the optimal solution is the progressive expansion of domain method. Consequently, the problem addressed here adopts a performance index that includes penalties for deviation from a specified state and for deviations from some predefined value of the control variable.

$$J = min\int_{t_0}^{t_f}\Big(K_h\big(h(t) - h(T)\big)^2 + K_u\big(u(t) - u(T)\big)^2\Big)\,dt \quad (5)$$

$$J = min\int_{t_0}^{t_f}\varphi(h(t), u(t), t)\,dt; \quad (6)$$

Subject to:

$$\dot{h}(t) = \boldsymbol{f}(h(t), u(t),\, t)$$
$$h(t_0) = h_0 \quad (7)$$

$$h(t_f) = h(T) \quad (8)$$
$$t_0 \le t \le t_f$$

where φ is a function, K_h and K_u are non-zero positive constants.

The (5) can be written as an augmented optimization problem by introducing a Lagrange multiplier $\lambda(t)$ as follows:

$$J = \int_{t_0}^{t_f} \left(\varphi(h(t), u(t), t) + \lambda(t)\left(f(h(t), u(t), t) - \dot{h}(t)\right)\right) d \quad (9)$$

applying the method of Pontryagin, a Hamiltonian function $H(h(t), u(t), \lambda(t), t)$ is defined as:

$$H = \varphi(h(t), u(t), t) + \lambda(t) f(h(t), u(t), t) \quad (10)$$

$$J = \int_{t_0}^{t_f} \left(H - \lambda(t)\dot{h}(t)\right) dt \quad (11)$$

applying the Euler-Lagrange equation to the Hamiltonian gives:

$$\frac{\partial H}{\partial h(t)} - \frac{d}{dt}[\lambda(t)] = 0$$
$$\dot{\lambda}(t) = -\frac{\partial H}{\partial h(t)} \quad (12)$$

$$\frac{\partial H}{\partial \lambda(t)} = f(h(t), u(t), t) - \dot{h}(t) = 0 \quad (13)$$

$$\dot{h}(t) = f(h(t), u(t), t) \quad (14)$$

$$\frac{\partial H}{\partial u(t)} = 0 \quad (15)$$

therefore,

$$H = K_h\left(h(t) - h(T)\right)^2 + K_u\left(u(t) - u(T)\right)^2 + \lambda(t)\left(-n_j \alpha h^{(1/2)}(t) + \mu\, u(t)\right)$$
$$\dot{h}(t) = -n_j \alpha h^{(1/2)}(t) + \mu\, u(t) \quad (16)$$

$$\dot{\lambda}(t) = -[2K_h\left(h(t) - h(T)\right) - \frac{1}{2}\lambda(t)\left(n_j \alpha h^{(-1/2)}(t)\right)] \quad (17)$$

$$0 = 2K_u\left(u(t) - u(T)\right) + \mu\lambda(t) \quad (18)$$

and the boundary condition,

$$h(t_0) = h_0$$
$$\lambda(t_f) = 1 \quad (19)$$

From the result above, the determination of the optimal control requires the solution of a two-point boundary value problem (TPBVP) since the initial conditions of the system are specified at the initial time and the value of the Lagrange multiplier or co-state is specified the terminal point. Meanwhile, the unknown control is related to the state and co-state through the optimality condition. From (18), $u(t)$ can be expressed as:

$$u(t) = -\frac{\mu\lambda(t)}{2\boldsymbol{K_u}} + u(T) \quad (20)$$

substituting (20) into (16) leads to two sets of first-order differential equations with split boundary conditions.

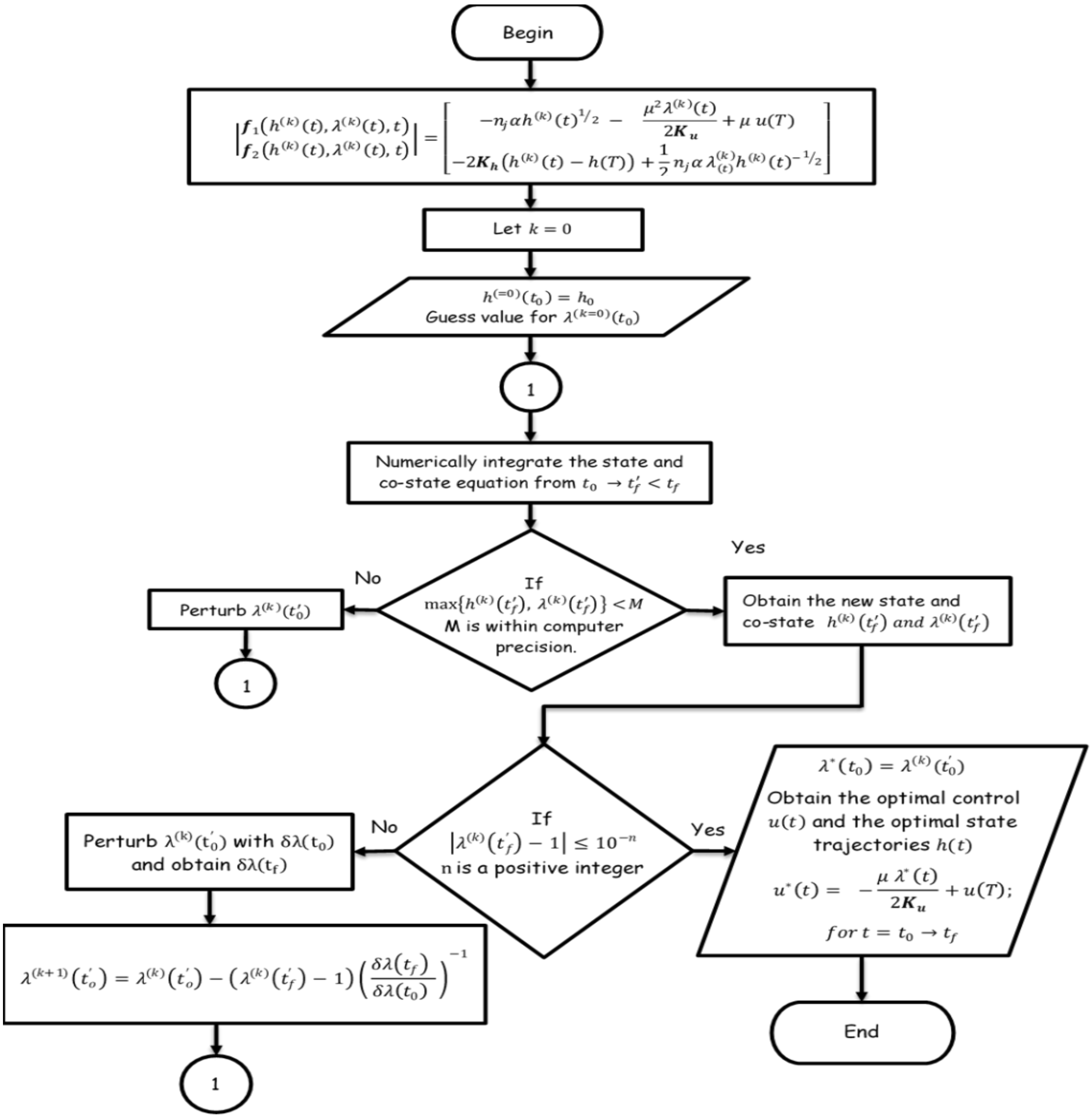
$$\dot{h}(t) = -n_j \alpha h^{(1/2)}(t) - \mu\left(\frac{\mu\lambda(t)}{2\boldsymbol{K_u}} + u(T)\right) \quad (21)$$
$$\dot{\lambda}(t) = -2\boldsymbol{K_h}(h(t) - h(T)) + \frac{1}{2}\lambda(t)\left(n_j \alpha h^{(-1/2)}(t)\right)$$
$$h(t_0) = h_0 \quad \text{and} \quad h(t_f) = unknown$$

$$\lambda(t_0) = unknown \quad \text{and} \quad \lambda(t_f) = 1 \quad (22)$$

2.1. Computation of the optimal control by a progressive domain expansion method (PDEM)

In (21) and (22) provide the basis for the computation of optimal control that consists of two nonlinear differential equations with split boundary conditions. The state equation is specified at the initial time while the co-state equation has its condition specified at the final time. The two-point boundary value problem (TPBVP) has attracted considerable attention in the past century. While many methods are found in the literature, one method that was often proposed but seldom used because of computational difficulties is the shooting technique. It is attractive because it involves guessing values for the missing initial conditions which in principle ought to be determined by one of several possibilities, the nature of the co-state equations lead to a rapid growth of the initial value problem that the computed values soon lose relationship with the problem since errors in computation are exponentially amplified by the system.

The PDEM is another modification proposed and pre-tested by [26], whereby instead of partitioning the domain [0, T], the final domain boundary is adjusted in such a manner that the initial guess results still retain a semblance with the original problem and the growing equations are bound so that the correct initial value problem is solved assuming the pseudo-domain $[0, T_k]$ where T_k is determined on the fly. In the next iteration, the initial guess for the missing boundary condition assumes the value that would have solved the problem for the pseudo-domain. Meanwhile, the problem is solved beyond T_k again until the growth begins to cause concern. The new T_{k+1} is thus defined and a new correction made so that the corresponding problem is solved. This process is repeated until $T_k = T_f$ in which case the value of the initial guess converges to the correct initial guess for the problem. The procedure is presented in the flowchart of Figure 2.



$$\begin{vmatrix} f_1(h^{(k)}(t), \lambda^{(k)}(t), t) \\ f_2(h^{(k)}(t), \lambda^{(k)}(t), t) \end{vmatrix} = \begin{bmatrix} -n_j \alpha h^{(k)}(t)^{1/2} - \frac{\mu^2 \lambda^{(k)}(t)}{2K_u} + \mu\, u(T) \\ -2K_h(h^{(k)}(t) - h(T)) + \frac{1}{2} n_j \alpha \lambda^{(k)}_{(t)} h^{(k)}(t)^{-1/2} \end{bmatrix}$$

$$\lambda^{(k+1)}(t'_o) = \lambda^{(k)}(t'_o) - (\lambda^{(k)}(t'_f) - 1)\left(\frac{\delta\lambda(t_f)}{\delta\lambda(t_0)}\right)^{-1}$$

Figure 2. Flow chart for the solution of optimal control canonical equations by a PDEM

3. RESULTS AND ANALYSIS

It is better to define a presentation format that reflects important features of the trajectory, many of which are deduced but not definitive by themselves in assessing a given trajectory. This leads to a very important matter of how to use optimal control for operational purposes. The notations for specifying operating conditions were formulated as follows: (number of operating machines, starting head (m), duration (hr), constraints on maximum inflow). This notation would be used in the presentation of results.

3.1. Case 1: (5,25.8,24,u(T) unconstrained)

Figure 3 presents the result for an indirect optimal control with the operational conditions of the machines and the initial value of the head at JHEPS specified as $(5{,}25.8{,}24, u(T)\ unconstrained)$. The operating condition implies that given that 5 turbo-alternators are running, while the operating head is 25.8 m. It is desired that the operating head increases to the nominal value $26.1m$ in $24\ hr$ $(86400s)$. The control problem is the determination of the inflow required to achieve this stated objective. The optimal control system generated a control law of (23);

$$u(t) = -1 \times 10^{-11}t^3 + 2 \times 10^{-6}t^2 - 0.1182t + 4715.9 \qquad (23)$$

The control started from a value around 4605 m^3/s at t_0 and decreases gradually to zero at t_f. Because the maximum control is not constrained, the trajectory of the operating head could not reach the terminal value $h(T) = 26.1$, as $h(t_f) = 25.94$m. The operating head also rose to a peak value and decreases later. Hence this result is not satisfactory, the algorithm has to be modified.

3.2. Case 2: (3,25.8,24,u(T) unconstrained)

Case 2 considered a situation where the number of units in operation reduces to 3 machines and the operating head 25.8 m. A similar result to that of Figure 3 was observed and presented in Figure 4. It can be concluded that to use PDEM, a constraint in the form of penalty on the terminal control $u(t_f)$ can be incorporated into the algorithm.

$$u(t) = -9 \times 10^{-12}t^3 + 2 \times 10^{-6}t^2 - 0.1165t + 4143.5 \qquad (24)$$

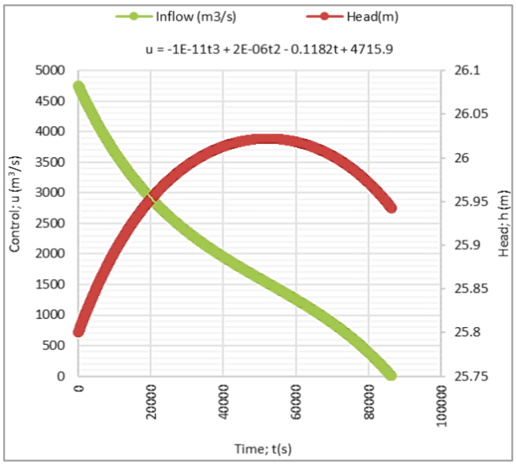


Figure 3. Optimum response $(5, 25.8, 1, u(T)\ Unpenalized)$

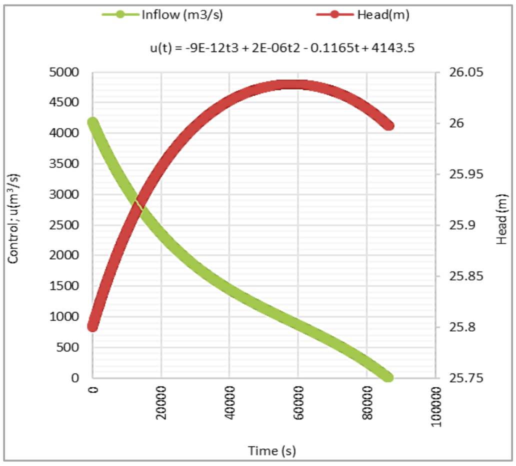


Figure 4. Optimum response (3, 25.8, 1, u(T) Unpenalized)

3.3. Case 3: (5,25.8,1,u(T) = 1800 m³⁄s)

Case 3 presents a similar condition to that of case 1 but with a constraint on the final control. This implies that the final control cannot decrease to zero, but a finite value specified as $u(T)$. The procedure for the selection of suitable value for $u(T)$ can be found in [5]. Figure 5 shows a better result of which the head moves from an initial value $h(0) = 25.8\ m$, to a final value $h(T) = 26.1\ m$ without an overshoot, as experienced in the direct optimal control. The optimal control defined as (25) starts around $4327.4\ m^3/s$ and ends at $1800\ m^3/s$.

$$u(t) = 5 \times 10^{-07}t^2 - 0.0702\,t + 4327.4 \qquad (25)$$

3.4. Case 4: (5,25.8,2,u(T) = 1800 m³/s)

A situation can be considered where the time limit for which optimal control is required increases to two days (48 hr.). It can be observed from Figure 6 that the PDEM could determine the solution as well except. In case 3, $u(0)$ started with $4327.4\, m^3/s$ but reduces slightly to $4270.1\, m^3/s$ in case 4, this implies that the time does not have significant effect on the starting control $u(0)$. The control law, in this case, is expressed as (26).

$$u(t) = -2 \times 10^{-12} t^{3} + 6 \times 10^{-07} t^{2} - 0.0689t + 4270.1 \tag{26}$$

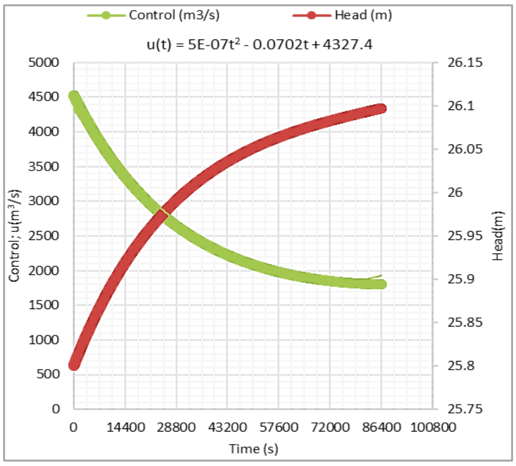


Figure 5. Optimum response $(5, 25.8, 1, u(T) = 1800\ m^3/s)$

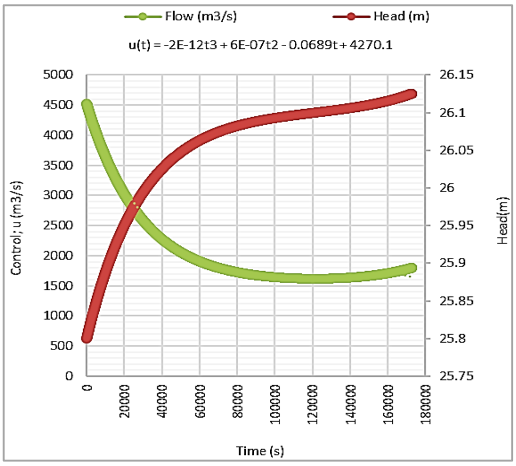


Figure 6. Optimum response $(5, 25.8, 2, u(T) = 1800\ m^3/s)$

3.5. Approximate optimal control: $(5, 25.8, 1, u(T) = 1800\ m^3/s)$

Consider the realization of the physical controller or the release of inflow from KHEPS, the infinite dimensional optimal control generated in case 1 to case 4 may not be easy to implement. An approximate optimal control may be necessary such that the inflow is released systematically withing every 6 hrs. This was considered for case 1, such that the average control with every 6 hrs was determined, the control is presented in Figure 7, where $u1 = 3648.20\ m^3/s$, $u2 = 2544.90\ m^3/s$, $u3 = 2040.46\ m^3/s$, $u4 = 1837.07\ m^3/s$. To show the performance of this approximation, the operating head trajectory generated is presented in Figure 8 to be satisfactory.

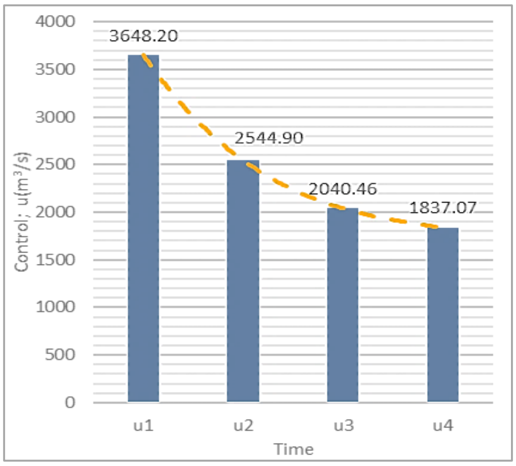


Figure 7. Approximate optimal control $(5, 25.8, 1, u(T) = 1800\ m^3/s)$

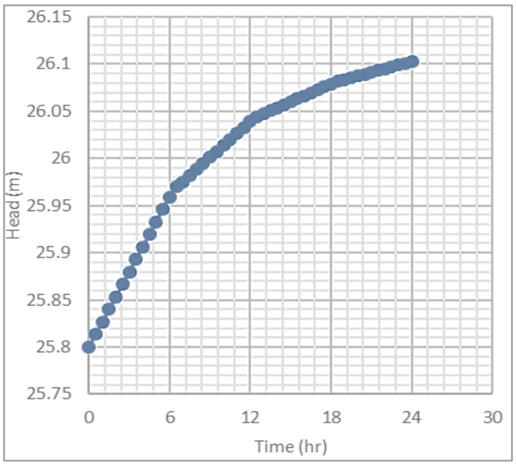


Figure 8. Head trajectory with approximate optimal control $(5, 25.8, 1, u(T) = 1800\ m^3/s)$

4. CONCLUSION

This paper proposed a progressive expansion of domain technique as a numerical approximation to the solution of an optimal control problem involving the regulation of the operating head of JHEPS. The solution was necessary knowing that the resulting two-point boundary value problem imposes a limitation in the use of indirect optimal control. From the result, the technique does not require a sophisticated initial guess like the normal shooting techniques and it converges faster than results expected from the nonlinear programming approach. The algorithm is therefore recommended for use in system studies, management and complete design of an electronic controller for the station.

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