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Formal expansion method for solving an electrical circuit model

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Abstract
We investigate the validity of the formal expansion method for solving a second order ordinary differential equation raised from an electrical circuit problem. The formal expansion method approximates the exact solution using a series of solutions. An approximate formal expansion solution is a truncated version of this series. In this paper, we confirm using simulations that the approximate formal expansion solution is valid for a specific interval of domain of the free variable. The accuracy of the formal expansion approximation is guaranteed on the time-scale 1.

Keywords: damped oscillation, electrical circuit, formal expansion, van der pol equation, vibration model

1. Introduction

Mathematics and its programming have played important roles in solving as well as designing experiments of electrical engineering problems, for example, see the work of Sutikno et al. [1-4]. To be specific, in this paper we consider electrical circuit problems. Problems in electrical circuits are often modelled into differential equations. One of the models is called the van der Pol equation. This equation is due to the Dutch physicist Balthasar van der Pol in around 1920 to describe oscillations in a triode-circuit [5]. In a specific situation with small source in oscillations, the van der Pol equation becomes a vibration model with a linear friction term. In this paper we solve the vibration model with a linear friction term, which is a modification of the van der Pol equation, using the formal expansion method.

Previous research has been conducted by a number of authors relating to the van der Pol equation [5-8] in physics [9-10], biology [11], economics [12], etc. [13-15]. Amongst them, Verhulst [5] provided a theorem about the order of accuracy of the formal expansion solution with respect to the perturbation factor in the damping term. Nevertheless, it has not been confirmed computationally when we use this method to solve the vibration model with a linear damping (friction term), especially the validity of the method relating to the interval of the free variable. Therefore, this paper shall fill this gap of research, that is, we shall validate of the formal expansion method computationally. The rest of this paper is written as follows. We provide the mathematical model and method in section 2. After that we present our research results and discussion in section 3. The paper is concluded with some remarks in section 4.

2. Mathematical Model and Method

The van der Pol equation, as the considered mathematical model, is

$$\ddot{x} + x = \mu(1 - x^2)\dot{x}$$

where μ is a positive constant [5]. When the factor $\mu(1 - x^2)$ is replaced by $-\varepsilon$, where ε is a small positive constant, the model becomes

$$\ddot{x} + x = -\varepsilon\dot{x}$$

which is valid for $x > 1$ or $x < -1$. This model is the vibration model with a linear friction term.

The core property in the formal expansion method is given in a theorem as follows due to Verhulst [5]. We consider the initial value problem

$$\dot{x} = f_0(t,x) + \varepsilon f_1(t,x) + \cdots + \varepsilon^m f_m(t,x) + \varepsilon^{m+1} R(t,x,\varepsilon)$$

where $x(t_0) = \eta$ and $|t - t_0| \leq h$, $x \in D \subset R^n$, $0 \leq \varepsilon \leq \varepsilon_0$. Here η is a constant, h is a positive constant, D is a domain in the n dimension, and ε_0 is a positive constant. We assume that in this domain all functions involved in the problem are infinitely many differentiable. Then the formal expansion

$$x_0(t) + \varepsilon x_1(t) + \cdots + \varepsilon^m x_m(t)$$

with $x_0(t_0) = \eta$, $x_i(t) = 0, i = 1, \dots, m$ approximates the exact solution $x(t)$ with the property

$$\left\| x(t) - \left(x_0(t) + \varepsilon x_1(t) + \cdots + \varepsilon^m x_m(t)\right)\right\| = 0(\varepsilon^{m+1})$$

on the time-scale 1. This means that the formal expansion is of the $(m + 1)$th order of accuracy.

3. Results and Discussion

For the convenience of writing and in order to be consistent with our references (such as Verhulst [5]), we consider the model

$$\ddot{x} + x = -2\varepsilon\dot{x}$$

suppose the initial conditions are $x(0) = a$ and $\dot{x}(0) = 0$. The exact solution to this problem is

$$x(t) = ae^{\varepsilon t}\cos\left(\sqrt{1 - \varepsilon^2}\,t\right) + \varepsilon\frac{a}{\sqrt{1-\varepsilon^2}}e^{-\varepsilon t}\sin\left(\sqrt{1 - \varepsilon^2}\,t\right)$$

substituting

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 \dots$$

into the model, we obtain

$$\ddot{x}_0 + x_0 = 0,$$
$$\ddot{x}_n + x_n = -2\dot{x}_{n-1}, \;\; n = 1,2, \dots$$

now we put

$$x_0(0) = a, \;\; \dot{x}_0(0) = (0)$$
$$x_n(0) = 0, \;\; \dot{x}_n(0) = 0, n = 1,2, \dots.$$

we obtain

$$x_0(t) = a\cos t$$
$$x_1(t) = a\sin t - at\cos t$$

therefore, our solution based on the formal expansion is

$$x(t) = a\cos t + a\varepsilon(\sin t - t\cos t) + \varepsilon^2 \dots$$

that is, the first order formal solution is

$$y_1(t) = a\cos t$$

the second order formal solution is

$$y_2(t) = a\cos t + a\varepsilon(\sin t - t\cos t)$$

Remark: We choose to consider this problem, because this problem has an exact solution. We intentionally use the exact solution to verify the validity of formal expansion solutions. If the

formal expansion solutions are valid for solving problems having exact solutions, then we shall be sure to use the formal expansion method to solve problems with the exact solutions are not known. Note that in practice, exact solutions are generally not known. Now for numerical experiments, we take $a = 1$ and vary the values of ε. To get clear illustrations, we take $\varepsilon = 0.5$, 0.05, 0.025 respectively.

3.1. Simulation for Case $\varepsilon = 0.5$

For the first case, we take $\varepsilon = 0.5$. Figure 1 shows the exact solution, the first order formal expansion solution, and the second order formal expansion solution on the interval $0 \leq t \leq 1$. We observe that the second order solution approximates the exact solution better than the first order does in the domain $0 \leq t \leq 1$. However, if we extend the domain to be $0 \leq t \leq 10$, the second order solution behaves poorly and even worse than the first order solution, as given in Figure 2.
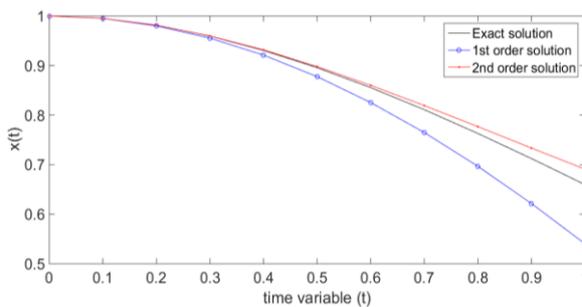

Figure 1. Exact, first order, and second order solutions for $\varepsilon = 0.5$ in domain $0 \leq t \leq 1$
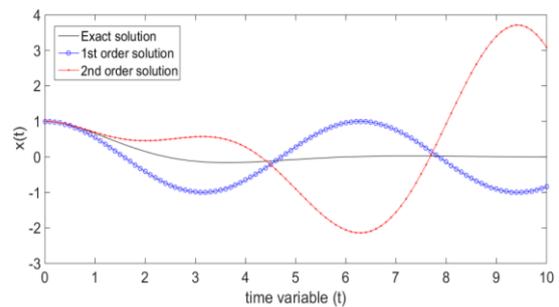
Figure 2. Exact, first order, and second order solutions for $\varepsilon = 0.5$ in domain $0 \leq t \leq 10$

3.2. Simulation for Case $\varepsilon = 0.05$

For the second case, we take $\varepsilon = 0.05$. Figure 3 shows the solutions on the interval $0 \leq t \leq 10$. Similar to the previous case, we observe that the second order solution approximates the exact solution better than the first order does in the domain $0 \leq t \leq 1$ and the extended domain $0 \leq t \leq 10$. However, if we extend the domain further to be $0 \leq t \leq 50$, the second order solution behaves worse than the first order solution, as illustrated in Figure 4.
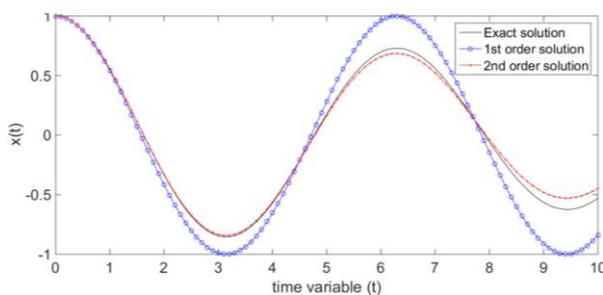

Figure 3. Exact, first order, and second order solutions for $\varepsilon = 0.05$ in domain $0 \leq t \leq 10$
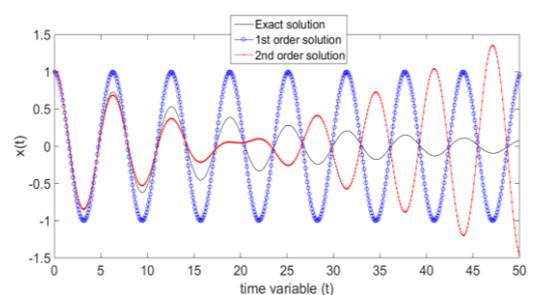
Figure 4. Exact, first order, and second order solutions for $\varepsilon = 0.05$ in domain $0 \leq t \leq 50$

3.3. Simulation for Case $\varepsilon = 0.025$

As the third case, we fix $\varepsilon = 0.025$. We plot the solutions on the interval $0 \leq t \leq 10$ as shown in Figure 5. Once again, we observe that the second order solution approximates the exact solution better than the first order does in the domain $0 \leq t \leq 1$ and the extended domain $0 \leq t \leq 10$. However, once again, if we extend the domain further to be $0 \leq t \leq 100$, the second order solution behaves worse than the first order solution, as illustrated in Figure 6.
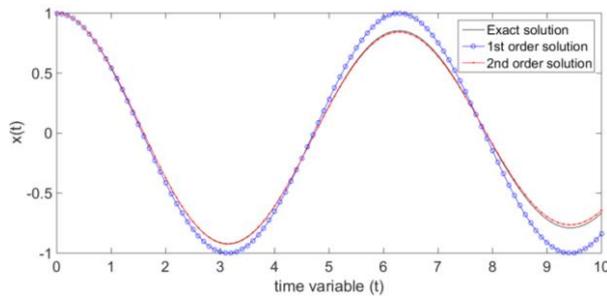
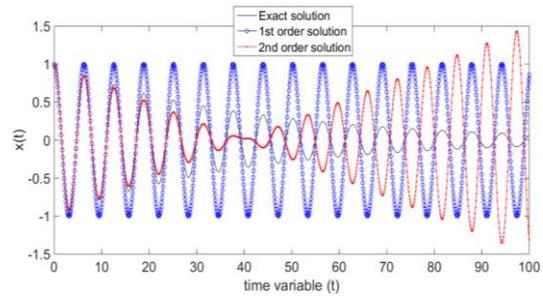Figure 5. Exact, first order, and second order solutions for $\varepsilon = 0.025$ in domain $0 \leq t \leq 10$

Figure 6. Exact, first order, and second order solutions for $\varepsilon = 0.025$ in domain $0 \leq t \leq 100$

3.4. Simulation for the Validity of Order of Accuracy

As we have mentioned in the mathematical method section, the formal expansion is guaranteed to be valid only on the time-scale 1. For any extension of the domain larger than $0 \leq t \leq 1$, the accuracy is not guaranteed. Obviously from the previous subsections (Subsections 3.1-3.3), we obtain that for an extended domain, the errors of the formal expansion solutions are indeed very large. In the present subsection we investigate the validity of the order of accuracy of the formal expansion. We limit our domain only on the interval of the time-scale 1. We take a discrete version of the time domain to be $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$. This means that we have discretised the time domain into 11 points. Error of an approximate solution is quantified as

$$Error = \frac{1}{N}\sum_{i=1}^{N}|x(t_i) - y(t_i)|$$

where N is the number of discrete time points t_i (in this case $i = 1, 2, 3, \ldots, N$ with $N = 11$), $x(t)$ is the exact solution, and $y(t)$ is the approximate solution. Furthermore, the order of accuracy is calculated as:

$$Order\ of\ accuracy = \frac{log\left(\frac{Error_j}{Error_{j+1}}\right)}{log\left(\frac{\varepsilon_j}{\varepsilon_{j+1}}\right)}$$

the order of accuracy is calculated based on the jth and the $(j + 1)$th simulations, respectively, using different values of ε. Our results of errors and orders of accuracy are summarised in Tables 1 and 2. Table 1 contains the errors of the first order formal solution with respect to varying ε on the time-scale 1. As ε tends to zero, the order of accuracy approaches 1. This is consistent with the theoretical background that the solution is of the first order. Table 2 summarises the errors of the second order formal solution with respect to varying ε on the time-scale 1. We find that as ε tends to zero, the order of accuracy approaches 2. This is consistent with the theory that as it is the second order formal expansion solution, the order of accuracy is 2 in the time-scale 1.

Table 1. Errors of the First Order Formal Solution with Respect to Varying ε on the Time-Scale 1

ε	Error	Order of accuracy
0.5	0.0351	-
0.25	0.0193	0.86
0.125	0.0101	0.93
0.0625	0.0052	0.96
0.03125	0.00	0.98

Table 2. Errors of the Second Order Formal Solution with Respect to Varying ε on the Time-Scale 1

ε	Error	Order of accuracy
0.5	0.007527	-
0.25	0.002037	1.89
0.125	0.000531	1.94
0.0625	0.000136	1.97
0.03125	0.000034	1.98

As final remarks, knowing the accuracy of the formal expansion method, we could extend the application of this method to solve other mathematical engineering problems, such as those studied by researchers in [16-26]. Possible other problems to be solved using the formal expansion method could be those in [27-37].

4. Conclusion

We have provided our research results on the formal expansion method for solving an electrical circuit model. The accuracy of the formal expansion is guaranteed on the time-scale 1. We have also confirmed the order of accuracy for the first and second order formal expansion solution using numerical experiments. We obtain that for the first order formal expansion solution, as the perturbation factor is halved, the error is also halved on the time-scale 1. For the second order formal expansion solution, as the perturbation factor is halved, the error is quartered on the time-scale 1. With these results, the formal expansion method could be used to solve other problems in electrical circuits for the time-scale 1. When the time-scale is not equal to 1, we may need to do re-scaling so that the time domain is on the time-scale 1. This could be a future research direction.

Acknowledgements

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