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Solving one-dimensional unconstrained global optimization problem using parameter free filled function method

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Abstract

It is generally known that almost all filled function methods for one-dimensional unconstrained global optimization problems have computational weaknesses. This paper introduces a relatively new parameter free filled function, which creates a non-ascending bridge from any local isolated minimizer to other first local isolated minimizer with lower or equal function value. The algorithm's unprecedented function can be used to determine all extreme and inflection points between the two considered consecutive local isolated minimizers. The proposed method never fails to carry out its job. The results of the several testing examples have shown the capability and efficiency of this algorithm while at the same time, proving that the computational weaknesses of the filled function methods can be overcomed.

Keywords: *curvature, filled function, global optimization, local optima, Newton's method*

1. Introduction

This article presents a new method for finding the global minimum of a non-convex function. A variety of fields, including engineering, operational research, finance and social sciences can be transformed as the optimization model where its objective function is non-convex. This non-convexity property makes the main reason a classical optimization method often fails to obtain a global minimizer (maximizer) [1-10]. The existing methods for finding a global minimizer provide the global descent [11, 12], the interval [13-17], the simulated annealing [18], the genetic algorithm [19], one-dimensional global optimization with Lipschitz conditions [20, 21], the filled function methods [4-6, 22-25] ideas and etc. Most of the filled function methods work quite different compared to the methods described in [11-22, 26-28]. This article will focus on the filled function methods and the comparison among them.

Suppose that x_k^* $(k = 0, \dots, m)$ are m isolated minimizers of f Four different parametric filled functions at x_k^* are defined by

$$F_G = F_G(x, x_k^*, r, \rho) = \frac{1}{r+f(x)} \exp\left(-\frac{\|x-x_k^*\|^2}{p^2}\right), \tag{1}$$

$$\begin{aligned} F_Z &= F_Z(x, x_k^*, r, \mu) \\ &= f(x_k^*) - \min\big(f(x_k^*), f(x)\big) - \rho\|x - x_k^*\|^2 + \mu\big\{\max\big(0, f(x) - f(x_k^*)\big)\big\}^2, \end{aligned} \tag{2}$$

$$F_{LS} = F_{LS}(x, x_k^*, \tau, \rho) = \eta(0.5\|x - x_0\|^2) + \varphi(\tau[f(x) - f(x_k^*) + p]), \text{ and} \tag{3}$$

$$F_X = F_X(x, x_k^*, a) = \frac{1}{\ln\left(1+f(x)-f(x_k^*)\right)} - a\|x - x_k^*\|^2, \tag{4}$$

are proposed in [4, 26, 29, 22] respectively where parameter a is defined by

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$$a = \frac{\xi|f'(x_s)|}{2|x_s - x_1^*|\left(1+f(x)-f(x_k^*)\right)\left(\ln\left(1+f(x)-f(x_k^*)\right)\right)^2}. \quad (5)$$

Unfortunately, the existing filled function methods [23, 26-29] can not solve the global optimization problems since:

a. cannot assure the existence of a better local minimizer in a lower basin [29, 30];
b. require the assumption that f has only a finite number of local minimizer which have different function values, i.e., $f(x_1^*) \neq f(x_2^*)$ if $x_1^* \neq x_2^*$;
c. difficult to adjust an appropriate parameter to satisfy the conditions of filled function;
d. iteratively updated the parameter;
e. can only obtain one global optimizer, and
f. contain exponential or logarithmic expressions in their forms which make a large amount of computation.

For filled function [24], its two parameters, one of which relies on the diameter of a bounded closed domain which contains all global minimizers, and the other on Lipschitz constant of f respectively. The parameter free filled function (PFFF) was initially introduced in [30-35]. A PFFF proposed by Ma et al. [34] is

$$F_M(x, x_k^*) = -sign\big(f(x) - f(x_k^*)\big)\arctan(\|x - x_k^*\|^2) \text{ where } sign(t) = \begin{cases} 1, & (t \geq 0) \\ -1, & (t < 0) \end{cases} \quad (6)$$

where this method also has weaknesses as the others.

Our new PFFF method (or simply IYRH's method) is based on PFFF [30-35] for global optimization of $f: D \subseteq R \to R$ where f satisfies the following seven assumptions:

A1. f is a trice continuously differentiable on D (or $f \in C^3(D)$)
A2. f has only a finite number of extreme and inflection points in D, and $f^{(n)}x_I \neq 0$ for $n \geq 3$ where x_I is an inflection point of f.
A3. $f^{(1)}$, $f^{(2)}$ and $f^{(3)}$ of f are Lipschitz-continuous with computable constants.
A4. $f(x) \to \infty$ as $|x| \to \infty$.
A5. For x_k^* ([36]), $f(x) - f(x_k^*) = 0$ yields at most two nearest points x_k^{Z-} and x_k^{Z+} located on the left and the right hand sides of x_k^*, respectively such that $f(x_k^{Z-}) = f(x_k^*) = f(x_k^{Z+})$ and $x \in [x_k^{Z-}, x_k^{Z+}]$.
A6. $f(x) > f(x_k^*)$ for $x \in (x_k^{Z-}, x_k^{Z+})$ and $f(x) = f(x_k^*)$ if $x = x_k^{Z-}$ or $x = x_k^{Z+}$.
A7. There exists only one x_I between two consecutive minimizer and maximizer of $f(x)$.

The reason why we need to solve one-dimensional multimodal function is described in many references cited in [20]. The needed is appeared in scientific and engineering applications especially in electrical engineering optimization problem. One of the important issues in global optimization is “the region of attraction” where its detail explanation can be seen in [3].

This paper is organized as follows. Section 2 describes the IYRH's function. Section 3 describes how to find all extreme and inflection points using the IYRH's function. Section 4 discusses the relationship between f and IYRH's function. In Section 5, the idea of curvature is described. Section 6 contains the convergence theorem. The numerical results of IYRH's algorithm will be presented in section 7. Comparison and discussion will be given in section 8. Section 9 contains the conclusion and the brief explanation on how this one-dimensional case can be extended to n-dimensional case.

2. A Relatively New Parameter Free Filled Function

In this section, the IYRH's will be derived. Definition 1 (One-Dimensional PFFF): Suppose that $f: [a, b] \subset R \to R$ satisfies A1–A7. A new $F(x, x_k^*)$ ($x \in [x_k^{Z-}, x_k^{Z+}]$) called IYRH's function of f at x_k^*, is defined by:

$$F(x, x_k^*) = \begin{cases} -\int_x^{x_k^*}\big(f(s) - f(x_k^*)\big)\,ds & (x_k^{Z-} \leq x \leq x_k^*) \\ -\int_{x_k^*}^{x}\big(f(s) - f(x_k^*)\big)\,ds & (x_k^* \leq x \leq x_k^{Z+}) \end{cases} \quad (7)$$

if $F(x, x_k^*)$ satisfies the following 3 conditions. C1. x_k^* is a local isolated maximizer of $F(x, x_k^*)$, C2. $F(x, x_k^*)$ has no stationary point in the interval $(x_k^{Z-}, x_k^*) \cup (x_k^*, x_k^{Z+})$, and C3.

If x_k^* is not a global minimizer of f, then x_k^{Z-} and x_k^{Z+} are the minimizer or stationary points of $F(x,x_k^*)$. It is enough to consider the second integration of (7) which can be rewritten as follows:

$$F(x,x_k^*) = -\int_{x_k^*}^{x}\big(f(s)-f(x_k^*)\big)\,ds, \qquad (x_k^* \le x \le x_k^{Z+}) \tag{8}$$

by using A5, the following results can be proved. Theorem 1: If 1) $f \in C^3(a,b)$; 2) $x_k^* \in [x_k^{Z-},x_k^{Z+}] \subseteq [a,b]$ and; 3) $F(x,x_k^*)$ is defined by (6), then x_k^* must be a local isolated maximizer of $F(x,x_k^*)$. Theorem 2: If the hypotheses of Theorem 1 are valid, then $F(x,x_k^*)$ does not have any stationary point in the interval $I_1 = \left\{x: f(x) > f(x_k^*), x \in (x_k^{Z-},x_k^{Z+})\backslash\{x_k^*\}\right\} = (x_k^{Z-},x_k^*) \cup (x_k^*,x_k^{Z+})$. By (8), for $x_1,x_2 \in [x_k^*,x_k^{Z+}]$ with $x_1 < x_2$, $F(x_1,x_k^*) - F(x_2,x_k^*) = \int_{x_1}^{x_2}\big(f(s)-f(x_k^*)\big)\,ds \ge 0$. Thus, $F(x,x_k^*)$ decreases over $[x_k^*,x_k^{Z+}]$. By similar argument, $F(x,x_k^*)$ increases over $[x_k^{Z-},x_k^*]$. Theorem 3: If the hypotheses of Theorem 1 are valid and x_k^* is not a global minimizer of $f(x)$, then x_k^{Z-} and x_k^{Z+} are the minimizer or stationary point of $F(x,x_k^*)$.

The obtaining of all the extreme and inflection points in every $[x_k^*,x_k^{Z+}]$ $(k = 0,1,2,\ldots,t < \infty)$ is an indicator that this algorithm never fail to obtain the global one. That is why it makes this method explores along the entire domain which very much different to other methods [4-6, 11-21].

We are not aware with the method in [37], which quiet similar with our method. Fortunately, our method has been published first as mentioned in [30-35]. However, we did not know how the authors [37] compute their integration. In IYRH's algorithm, the integration is never been computed as had been done in [30-35]. Therefore, IYRH's algorithm very much different compared with others.

3. Sequences of Extreme and Inflection Points

The IYRH's function $F(x,x_k^*)$ $(x \in [x_k^{Z-},x_k^{Z+}])$ has the following properties:

P1. $F(x,x_k^*)$ is concave downward at x_k^* and concave upward at both x_k^{Z-} and x_k^{Z+}.
P2. $F(x,x_k^*)$, $F^{(1)}(x,x_k^*)$, $F^{(2)}(x,x_k^*)$ and $F^{(3)}(x,x_k^*)$ are continuous.
P3. $F(x,x_k^*) < 0$ for $[x_k^{Z-},x_k^*] \cup [x_k^*,x_k^{Z+}]$ and $F(x_k^*,x_k^*) = 0$.
P4. $F(x,x_k^*)$ are increasing and decreasing over $[x_k^{Z-},x_k^*]$ and $[x_k^*,x_k^{Z+}]$ respectively.
P5. $F^{(1)}(x,x_k^*) > 0$ $\big(x \in (x_k^{Z-},x_k^*)\big)$ and $F^{(1)}(x,x_k^*) < 0)$ $\big(x \in (x_k^*,x_k^{Z+})\big)$ except at inflection points
P6. $F(x,x_k^*)$ has isolated minimizer or stationary point at x_k^{Z-} or x_k^{Z+}.

As an example, the graph of our PFFF for $\sin x + \sin(2x/3)$ can be plotted as in Figure 1. By P1, there exists at least two inflection points of $F(x,x_k^*)$ each lies in (x_k^{Z-},x_k^*) and (x_k^*,x_k^{Z+}). By P1-P6, Figure 1 and $[x_k^*,x_k^{Z+}]$ $([x_k^{Z-},x_k^*])$, the IYRH's function generates the sequence of:

$$x_0^*,\, x_1^*,\, x_2^*,\, \ldots,\, x_k^*,\, \ldots\ (x_0^*, x_{-1}^*, x_{-2}^*,\, \ldots,\, x_{-k}^*, \ldots) \tag{9}$$

using any suitable optimization tools except x_0^*, starting at:

$$x_0,\, x_0^{Z+},\, x_1^{Z+},\, x_2^{Z+} \ldots,\, x_k^{Z+},\, \ldots\ (x_0,\, x_0^{Z-},\, x_{-1}^{Z-},\, x_{-2}^{Z-}\, \ldots,\, x_{-k}^{Z-},\, \ldots) \tag{10}$$

respectively where as beginning x_0 is any given point for obtaining x_0^*. Therefore, we have 2 phases as follows:

Phase I : Starting at x_k $(k = 0,1,2,\ldots)$. Minimize $f(x)$ to obtain isolated minimizer x_k^*.
Phase II : $F(x,x_k^*)$ is constructed to find x_k^{Z+}. Replace x_k with x_k^{Z+}. Restart Phase 1.

However, it is not easy to obtain (9) and (10) since $F(x,x_k^*)$ contains at least 1 inflection point in $[x_k^*,x_k^{Z+}]$ and also in $[x_k^{Z-},x_k^*]$. The analytical existence of inflection points is shown as follows.

a. $F^{(1)}(x,x_k^*) = -\big(f(x)-f(x_k^*)\big)$ $(x_k^* \le x \le x_k^{Z+})$ and $F^{(2)}(x,x_k^*) = -f^{(1)}(x)$ $(x_k^* \le x \le x_k^{Z+})$. If $F^{(2)}(x,x_k^*) = 0$, then $f^{(1)}(x) = 0$. Therefore, the solution of $F^{(2)}(x,x_k^*) = 0$ becomes the critical point of $f(x)$.
b. $F^{(3)}(x,x_k^*) = -f^{(2)}(x)$ $(x_k^* \le x \le x_k^{Z+})$, if $F^{(3)}(x,x_k^*) = 0$, then $f^{(2)}(x) = 0$. Therefore, by A2, the solution of $F^{(3)}(x,x_k^*) = 0$ (the critical point of $F^{(2)}(x,x_k^*)$) becomes the inflection point of $f(x)$ but this solution becomes the critical point (maximizer or minimizer) of $F^{(2)}(x,x_k^*)$.

Therefore, the isolated extreme points i.e. $x_{M,k}^{(1)}$ (local maximizer) and $x_{m,k}^{(1)}$ (local minimizer) of $f(x)$ as shown in Figure 2, are inflection points of $F(x,x_k^*)$ where $^{(1)}$ the superscript of $x_{M,k}^{(1)}$ and $x_{m,k}^{(1)}$, denotes the first inner iteration in the interval $[x_k^*, x_k^{Z+}]$.

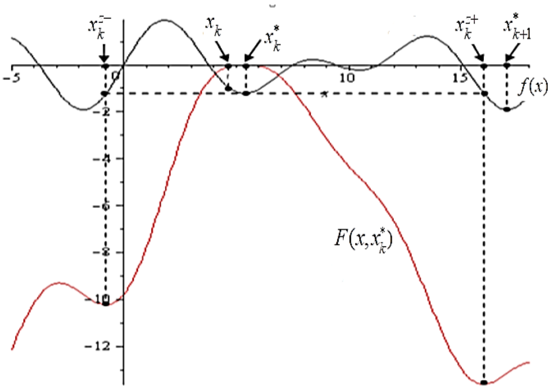


Figure 1. $F(x,x_k^*)$ $x \in [x_k^{Z-}, x_k^{Z+}]$) for $\sin x + \sin(2x/3)$

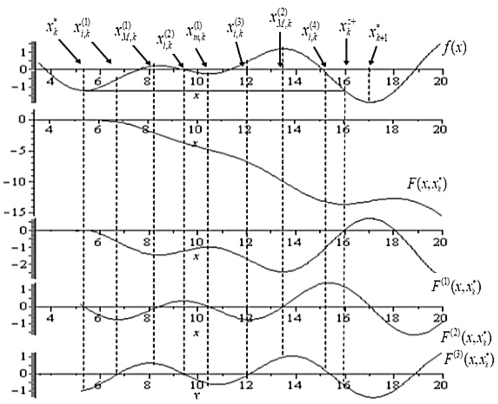


Figure 2. Relationship between $f(x)$, $F(x,x_k^*)$, $F^{(1)}(x,x_k^*)$, $F^{(2)}(x,x_k^*)$ and $F^{(3)}(x,x_k^*)$

Note that there might exist more than one inner iteration in the interval $[x_k^*, x_k^{Z+}]$ and this will happen when more than two inflection points occurred in $[x_k^*, x_k^{Z+}]$. Therefore for Phase II, we need to analyse the behaviour of $F(x,x_k^*)$. Since x_k^* cannot be used blindly to minimize the filled function, then in the phase II, a $\delta > 0$ must be chosen such that $x_k^* + \delta$ can be safely utilized to minimize $F(x,x_k^*)$. For handling these difficulties, consider the relationship between f, F and $F^{(2)}$ as illustrated in Figure 2. Since Newton's method [38] sometimes fails to converge to x_k^{Z-} or x_k^{Z+}, we need IYRH's function method to handle it. From Figure 2 and the above discussion, it is clear that all minimizers and maximizers of $F^{(2)}(x,x_k^*)$ become the inflection points of $f(x)$, and all the roots of $F^{(2)}(x,x_k^*)$ become minimizers or maximizers of $f(x)$. These special properties are only possessed by IYRH's function.

4. Computation of the Inflection Points

Based on the discussion in Section 3, we have proved the following theorems: Theorem 4: If the hypotheses of Theorem 1 are valid, then the solution of $F^{(2)}(x,x_k^*) = 0$ becomes the critical point of $f(x)$. Theorem 5: If the hypotheses of Theorem 1 are valid, then the critical point of $F^{(2)}(x,x_k^*)$ becomes the inflection points of $f(x)$.

By A5 and Figure 2, $x_{i,k}^{(1)}$, $x_{M,k}^{(1)}$ and $x_{m,k}^{(1)}$ are the first isolated inflection, maximum and minimum points of $f(x)$ respectively, $x_{i,k}^{(2)}$is the second inflection point of $f(x)$ found after x_k^* where $x_{i,k}^{(1)} < x_{M,k}^{(1)} < x_{i,k}^{(2)} < x_{m,k}^{(1)}$, and it might continue with another sequence of extreme and inflection points until x_k^{Z+} such that $f(x_k^{Z+}) = f(x_k^*)$ and $f(x_k^*) < f(x)$ $\left(x \in (x_k^*, x_k^{Z+})\right)$. In order to guarantee no extreme or inflection points of $f(x)$ missed during the computation, the outer and inner iterations are used over $[x_k^*, x_k^{Z+}]$. In inner iteration, $F^{(3)}(x,x_k^*)$ and $F^{(2)}(x,x_k^*)$ are used to compute inflection and extreme points of $f(x)$ respectively whereas in outer iteration, $f(x)$ is minimized or solve $F^{(1)}(x,x_k^*) = 0$ to obtain x_k^{Z+}. The following steps implement those both inner and outer iterations:

Outer Iteration

Step 1 : construct $F(x,x_k^*)$ at x_k^*.

Inner Iteration

Step 2 : Solve $F^{(3)}(x,x_k^*) = 0$ by Newton's method for inflection point of $f(x)$ nearest to x_k^*.

Step 3 : Solve $F^{(2)}(x,x_k^*) = 0$ by Newton's method for isolated maximizer of $f(x)$.

Step 4 : Solve $F^{(3)}(x,x_k^*) = 0$ by Newton's method for next inflection point of $f(x)$.

Step 5 : Solve $F^{(2)}(x,x_k^*) = 0$ by Newton's method for isolated minimizer of $f(x)$.

Step 6 : If $f(x) > f(x_k^*)$and $x < x_k^{Z+}$ then repeat Step 2 - Step 5 else solve $F^{(1)}(x, x_k^*) = 0$ by Newton's method for x_k^{Z+}such that $f(x_k^{Z+}) = f(x_k^*)$.
Step 7 : Use x_k^{Z+} to yield x_{k+1}^*. $k \coloneqq k + 1$. Go to Step 1 if $x_{k+1}^* < b$.

5. Convergence with Curvature

The curvature [39] and radius of curvature are defined by:

$$\kappa(x) = \left|\frac{d\varphi}{dx}\right| = \frac{|d^2x/dx^2|}{[1+(dy/dx)^2]^{3/2}} \text{ and } \rho(x) = \frac{1}{\kappa(x)}$$

respectively. Basically, to make Newton's method converges to x_*, the solution of $f(x) = 0$, we need an initial estimation which closes enough to x_*. Assign the radius of curvature of $f(x)$ to ρ, therefore $x_k^* + \rho$ becomes the initial best estimator for Newton's method to solve $f(x) = 0$. We will prove that $\eta = |x_k^* + \rho - x_*|$ is the radius of the largest interval around x_* such that the Newton's method converges to $x_* \in (x_* - \eta, x_* + \eta)$. However, we will need the following definition.

Definition 2 [40]: The function $f: D \subset R \to R$ is Lipschitz continuous function with constant γ in D, written $f \in Lip_\gamma(D)$, if for every $x, y \in D$, $|f(x) - f(y)| \le \gamma|x - y|$.

For the convergence of Newton's method, we need $f^{(1)} \in Lip_\gamma(D)$ which had been shown in [40].

Lemma 1 [40] : If 1) $f: D \subset R \to R$ for an open interval D; 2) $f^{(1)} \in Lip_\gamma(D)$, then for any $x, y \in D$, $\left|f(y) - f(x) - f^{(1)}(x)(y - x)\right| \le \gamma(y - x)^2/2$.

For most problems, Newton's method will converge q-quadratically to its root [40].

Theorem 6 [40]: If 1) $f: D \subset R \to R$ for an open interval D; 2) $f^{(1)} \in Lip_\gamma(D)$ 3) for some $\beta > 0$, $\left|f^{(1)}(x)\right| \ge \beta$ $(x \in D)$; 4) $f(x) = 0$ has a solution $x_* \in D$, then there is some $\eta > 0$ such that if $|x_0 - x_*| < \eta$, then $\{x_n\}$ generated by $x_{n+1} = x_n - \left(f(x)/f^{(1)}(x)\right)$ $(n = 0,1,2, \dots)$ exists and converges to x_*. Furthermore,$|x_{n+1} - x_*| \le (\gamma/2\beta)|x_n - x_*|^2$ $(n = 0,1,2, \dots)$.

Now, we prove that $\hat{\eta} = |x_1^* + \hat{\rho} - x_*|$, the radius of the largest interval around the solution of $f^{(1)}(x) = 0$ holds Theorem 6. The similarity proof is applied for $F^2(x, x_k^*)$. Theorem 7: If 1) $f: D \subset R \to R$ is an objective function; 2) x_1^* is a local isolated minimizer of $f(x)$; 3) $f^{(1)}: D \subset R \to R$ and $f^{(2)} \in Lip_\gamma(X)$ for $X \subseteq D$; 4) for some $\rho > 0$, $\left|f^{(2)}(x)\right| \ge \rho$ for every $x \in D$; 5) $f^{(1)}(x) = 0$ has a solution $x_* \in D$, then there is some $\eta > 0$ such that if $|x_0 - x_*| < \eta$, then the sequence $\{x_n\}$ generated by $x_{n+1} = x_n - \left(f^{(1)}(x_n)/f^{(2)}(x_n)\right)$ $(n = 0,1,2, \dots)$ exists and converges to x_*. Furthermore, $|x_{n+1} - x_*| \le (\gamma/2B)|x_n - x_*|^2$ $(n = 0,1,2, \dots)$.

6. Convergence of the IYRH's Algorithm

By A2, IYRH's algorithm actually generates (9) and (10) according to the following pattern:

$$\begin{tikzcd}
x_0^* \arrow[dr] & x_1^* \arrow[dr] & x_2^* & & x_{k-1}^* \arrow[dr] & x_k^* & & x_{n-1}^* \arrow[dr] & x_n^* \\
x_0 \arrow[u] & x_0^{Z+} \arrow[u] & x_1^{Z+} \arrow[u] & \cdots & x_{k-2}^{Z+} \arrow[u] & x_{k-1}^{Z+} \arrow[u] & \cdots & x_{n-2}^{Z+} \arrow[u] & x_{n-1}^{Z+} \arrow[u]
\end{tikzcd}$$

which satisfy $f(x_0^*) \ge f(x_1^*) \ge \dots \ge f(x_k^*) \ge \dots \ge f(x_{n-1}^*) \ge f(x_n^*)$ where x_0 is any given point in the considered interval. Therefore, IYRH's algorithm generates a finite sequence $[x_0^*, x_0^{Z+}]$, $[x_1^*, x_1^{Z+}], \dots, [x_k^*, x_k^{Z+}], \dots, [x_{n-1}^*, x_{n-1}^{Z+}]$. Thus, by A2, IYRH's algorithm converges to x_n^* as a global minimizer. IYRH's algorithm also automatically generates at least a set of finite sequence of inflection, local isolated maximizers and isolated minimizers, $A_1 = \left\{x_{i,k}^{(1)}, x_{M,k}^{(1)}, x_{i,k}^{(2)}\right\}$ in every subinterval $[x_k^*, x_k^{Z+}]$ $(k = 0, 1, \dots, n)$ if exist where the superscript $^{(1)}$ on $x_{M,k}^{(1)}$ denotes the first number of local maximizer and subscript of A denotes the number of local maximizer in $[x_k^*, x_k^{Z+}]$. If it contains two local isolated maximizers, then it generates

$A_2 = \left\{x_{i,k}^{(1)}, x_{M,k}^{(1)}, x_{i,k}^{(2)}, x_{m,k}^{(1)}, x_{i,k}^{(3)}, x_{M,k}^{(2)}, x_{i,k}^{(4)}\right\}$ and so forth. However for A_2, the inflection point$x_{i,k}^{(4)}$ is option. Thus, the following theorem is proved.

Theorem 8 (Convergence Theorem): If 1) all the hypothesis of Theorem 6 and Theorem 7 are valid for f, $f^{(1)}$, $f^{(2)}$, $f^{(3)}$ and; 2) $F(x, x^*)$ is IYRH's function at x^*, the local isolated minimizer of f, then IYRH's algorithm converges to the right solution.

7. Numerical Results

The test examples are listed in Tables 1–3. In Table 1 where N, $f(x)$, D, v_g^* and m_k^* denote the number of function, the expression of the objective function, the domain, global minimum value and global minimizer respectively. The numerical results will be presented to compare the capability of the IYRH's method with two-parameter filled function methods [4, 26, 29], one-parameter filled function methods [6, 22, 28], and the PFFF method [34]. We also present the results for observing the sensitivity of IYRH's method due to different initial points. Therefore, the presentation is arranging into four categories.

Table 1. 20 Test Functions Cited from [41] for Minimization Problem (Original Results)

N	$f(x)$	D	v_g^*	m_g^*
1.	$\frac{1}{6}x^6 - \frac{52}{25}x^5 + \frac{39}{80}x^4 + \frac{71}{10}x^3 - \frac{79}{20}x^2 - x + \frac{1}{10}$	$[-1.5,11]$	-29763.233	10
2.	$\sin x + \sin(10/3)x$	$[2.7,7.5]$	-1.899599	5.145735
3.	$-\sum_{k=1}^{5} k\sin\left((k+1)x+k\right)$	$[-9.4,10]$ $[-9.4,10]$ $[-9.4,10]$	-12.03124 -12.03124 -12.03124	-6.7745761 -0.491391 5.791785
4.	$-(16x^2 - 24x + 5)e^{-x}$	$[1.9,3.9]$	-3.85045	2.868034
5.	$-(-3x + 1.4)\sin(18x)$	$[0,1.2]$	-1.48907	0.96609
6.	$(-x + \sin(x))e^{-x^2}$	$[-10,10]$	-0.824239	-0.679579
7.	$\sin(x) + \sin((10/3)x) + \ln x - 0.84x + 3$	$[2.7,7.5]$	-1.6013	5.19978
8.	$-\sum_{k=1}^{5} k\cos\left((k+1)x+k\right)$	$[-9.7,10]$ $[-9.7,10]$ $[-9.7,10]$	-14.508 -14.508 -14.508	-7.083506 -0.800321 5.48286
9.	$\sin x + \sin(2/3)x$	$[3,20]$	-1.90596	17.039
10.	$-x\sin x$	$[0,10]$	-7.91673	7.9787
11.	$-2\cos x - \cos 2x$	$[-1.57,6.28]$	-3	$4.76837e-009$
12.	$\sin^3 x + \cos^3 x$	$[0,6.28]$ $[0,6.28]$	-1 -1	π 4.712389
13.	$-x^{2/3} - (1-x^2)^{1/3}$	$[0.001,0.99]$	-1.5874	$1/\sqrt{2}$
14.	$-e^{-x}\sin 2\pi x$	$[0,4]$	-0.788685	0.224885
15.	$(x^2 - 5x + 6) \div (x^2 + 1)$	$[-5,5]$	-7.03553	-0.41422
16.	$2(x-3)^2 + e^{-x^2/2}$	$[-3,3]$	0.0111090	3
17.	$x^6 - 15x^4 + 27x^2 + 250$	$[-4,4]$ $[-4,4]$	7 7	-3 -3
18.	$\begin{cases} (x-2)^2 & x \le 3 \\ 2\ln(x-2)+1 & \text{(otherwise)} \end{cases}$		0	2
19.	$-\sin 3x + x + 1$	$[0,6.5]$	$0{,}467511$	$0{,}41032$
20.	$(-x + \sin x)e^{-x^2}$	$[-10,10]$	-0.0634905	-1.19514

For first category, the results of Table 4 shows that IYRH's algorithm can solve the global optimization problems listed in Table 1. In Table 4, k $(k \ge 0)$ is the number of outer iteration, j $(j \ge 0)$ is the number of inflection points $x_{i,k}^{(j)}$ and $x_{i,k}^{(j+1)}$ where i refers to the word "inflection", n is the number of local isolated maximizer $x_{M,k}^{(n)} \in [x_k^*, x_k^{Z+}]$ and minimizer $x_{m,k}^{(n)} \in [x_k^*, x_k^{Z+}]$ where M and m denote maximizer and minimizer respectively, x_k^* $(k \ge 0)$ is isolated minimizer and $x_k^{Z+}$and x_k^{Z-} $(k \ge 0)$ are points such that $f(x_k^{Z-}) = f(x_k^*) = f(x_k^{Z+})$.

In inner iteration, there are several cases that $x_{m,k}^{(n)}$ equals to $x_k^{Z+}$and x_{k+1}^* as shown in Table 4 for $k=2,3$ of example 3 ($N=3$), $k=3$ of example 8 ($N=8$), $k=1$ of example 12 ($N=12$) and $k=0$ of example 17 ($N=17$). For second category, Tables 5–8 compare IYRH's algorithm with New algorithm [42], the direct method [42] and Lagrange interpolation [43], using test functions in Table 1 [44] and 100 one-dimensional randomized test functions [45]. Table 5 shows the relative errors [42] of global minimum values and global minimizers of functions as shown in Table 1 obtained by IYRH's algorithm is better than Lagrange interpolant on 81 Chebyshev nodes [43]. Table 6 shows that a "fortune effect" does not happened to IYRH's algorithm when it is applied to the example given in Table 3 for $r=67$ and x_r^* is chosen randomly and differently where its graph is shown in Figure 3.

Table 2. 7 Test Functions for Comparison with Existed Filled Function Methods

N	$f(x)$, D, x_0
1	$f(x)=\sin(x)+\sin(10x/3)+\ln(x)-0.84x$
2	$f(x)=-\sum_{i=1}^{5}\sin\big((i+1)x+i\big)$
3	$f(x)=\frac{\pi}{n}\left\{k\sin^2(\pi y_1)+\sum_{i=1}^{n-1}[(y_1-A)^2(1+k\sin^2(\pi y_{i+1}))]+(y_n-A)^2\right\}$ $y_1=1+-0.25(x_i-1)$, $k=10$, $A=1$ and n denotes the dimensionality of the problem
4	$f(x)=\frac{\pi}{n}\left\{k\sin^2(\pi x_1)+\sum_{i=1}^{n-1}[(x_1-A)^2(1+k\sin^2(\pi x_{i+1}))]+(x_n-A)^2\right\}$ $k=10$, $A=1$ and n denotes the dimensionality of the problem
5	$f(x)=k\sin^2\pi l_0x_1+k_1\sum_{i=1}^{n-1}[(x_1-A)^2(1+k\sin^2\pi l_0x_1)]+k_1(x_n-A)^2(1-k_0\sin^2\pi l_0x_1)$ where the constants in this equation have been fixed as follows: $k_0=1$, $k_1=0.1$, $A=1$, $l_0=3$

Table 3. Test Functions for "Fortune Effect" of IYRH's Function [44]

r	f_r	D
$1,\dots,100$	$0.025(x-x_r^*)^2+\sin^2[(x-x_r^*)+(x-x_r^*)^2]+\sin^2(x-x_r^*)$	$[-5,5]$

For third category, Table 7 compares the results of IYRH's algorithm with Ma et al.'s filled function and Lucidi and Piccially's filled function for a set of 5 test examples in Table 2. For the last category, the results presented in Table 8 is used to observe the sensitivity of IYRH's algorithm due to three initial points using example 3 from Table 2. It is clear that IYRH's function can be used to solve the global optimization problems from any initial point.

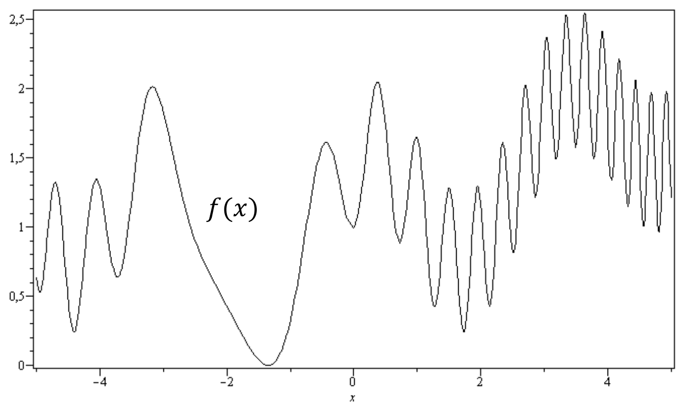


Figure 3. Graph of $f_{67}(x)$ one of the 100 one-dimensional randomized test functions

Table 4. Numerical Results of a Set of 20 Test Functions by IYRH's Algorithm

N	k	j/n	x_0	x_k^*	$x_{i,k}^{(j)}$	$x_{M,k}^{(n)}$	$x_{i,k}^{(j+1)}$	$x_{M,k}^{(n)}$	x_k^{Z+}
1	0	1/1	-1.5	-1.41421	-0.982448	-0.1	0.18688	0.5	
		3/2			1.06955	1.41421	8.0460129		2.04497
	1	1/1		10					
2	0	1/1	2.7	3.38725	3.78614	4.1966	4.68536		4.77334
	1	1/1		5.14574	5.6704	6.21731	6.60591	7.00015	
3	0	1/1	-9.4	-9.03744	-8.78099	-8.54977	-8.25976	-8.00868	-8.24149
	1	1/1		-8.00868	-7.6897	-7.39728	-7.09257	-6.77458	-6.90218
	2	1/1		-6.77458	-6.50838	-6.20297	-5.9699	-5.70624	
		3/2			-5.46785	-5.21159	-4.97098	-4.71981	
		5/3			-4.47764	-4.23167	-3.9846	-3.73921	
		7/4			-3.49188	-3.25263	-2.996	-2.75426	
		9/5			-2.4978	-2.26658	-1.97658	-1.72549	
		11/6			-1.40652	-1.1141	-0.809383	-0.491391	-0.491391
	3	1/1		-0.491391	-0.225195	0.0802188	0.313287	0.57695	
		3/2			0.815333	1.07159	1.3122	1.56337	
		5/3			1.80555	2.05152	2.29859	2.54398	
		7/4			2.7913	3.03056	3.28718	3.52893	
		9/5			3.78538	4.01661	4.30661	4.55769	
		11/6			4.87667	5.16909	5.4738	5.79179	5.79179
	4	1/1		5.79179	6.05799	6.3634	6.59647	6.86014	
		3/2			7.09852	7.35478	7.59539	7.84656	
		5/3			8.08873	8.3347	8.58177	8.82716	
		7/4			9.07449	9.31375	9.57037	9.81211	
4	0	1/1	1.9	2.86803					
5	0	1/1	0	0.0793517	0.15548	0.247978	0.314097	0.398387	
		3/2			0.44637	0.496343	0.569406	0.629167	
		5/3			0.721013	0.794718	0.887021		0.927292
	1	1/1		0.966086	1.05752	1.13904			
6	0	1/1	-10	-0.679579	2.98021e-009	0.679579	1.17698		
7	0	1/1	2.7	3.43923	3.78421	4.13614	4.6866		4.56652
	1	1/1		5.19978	5.66958	6.15443	6.60654	7.06776	
8	0	1/1	-9.7	-9.28634	-9.03059	-8.79406	-8.52587		-8.45402
	1	1/1		-8.29039	-7.98039	-7.70831	-7.39207		-7.31459
	2	1/1		-7.08351	-6.79634	-6.47857	-6.23259	-5.94894	
		3/2			-5.71688	-5.4614	-5.21924	-4.96318	
		5/3			-4.72391	-4.47753	-4.23112	-3.98396	
		7/4			-3.73827	-3.49725	-3.24447	-3.00316	
		9/5			-2.74741	-2.51088	-2.24269	-2.0072	
		11/6			-1.6972	-1.42513	-1.10889	-0.800321	-0.800321
	3	1/1		-0.800321	-0.513159	-0.195386	0.0505096	0.334244	
		3/2			0.566304	0.821784	1.06394	1.32	
		5/3			1.55927	1.80566	2.05197	2.29923	
		7/4			2.54492	2.78593	3.03872	3.28003	
		9/5			3.53578	3.77231	4.0405	4.27598	
		11/6			4.58598	4.85806	5.1743	5.48286	5.48286
	4	1/1		5.48286	5.77003	6.0878	6.33378	6.61743	
		3/2			6.84949	7.10497	7.34713	7.60319	
		5/3			7.84246	8.08884	8.33515	8.58241	
		7/4			8.8281	9.06912	9.3219	9.56321	
		9/5			9.81896				
9	0	1/1	3	5.36225	6.73129	8.39609	9.42478	10.4535	
		3/2			12.1183	13.4873	15.3753		15.9845
	1	1/1		17.0392	18.8496				
10	0	1/1	0	2.02876	3.6436	4.91318	6.57833		6.56409
	1	1/1		7.97867	9.62956				
11	0	1/1	-1.57	4.76837e-009	0.935929	2.0944	2.57376	3.14159	
		3/2			3.70942	4.18879	5.34726		
12	0	1/1	0	0.785398	1.20593	1.5708	2.35619		1.98146
	1	1/1		3.14159	3.50646	3.92699	4.34753	4.71239	4.71239
	2	1/1		4.71239	5.49779				
13	0	1/1	0.001	0.707107					
14	0	1/1	0	0.22488	0.449761	0.72488	0.949761	1.22488	
		3/2			1.44976	1.72488	1.94976	2.22488	
		5/3			2.44976	2.72488	2.94976	3.22488	
		7/4			3.44976	3.72488	3.94976		
15	0	1/1	-5	-0.414214	0.267949	2.41421	3.73205		
16	0	1/1	-3	3					
17	0	1/1	-4	-3	-2.38396	-1	-0.56277	1.054185e-008	
		3/2			0.56277	1	2.38396	3	3
		5/3		3					
18	0	1/1	0	2	(3 , 1)				
19	0	1/1	0	0.41032	1.0472	1.68408	2.0944	2.50471	
		3/2			3.14159	3.77847	4.18879	4.59911	
		5/3			5.23599	5.87287	6.28319		
20	0	1/1	-10	-1.19514	-0.69004	-1.61476e-010	0	1.61476e-010	
		3/2			0.69004	1.19514	1.69015		

Table 5. The Comparison of Relative Errors of Minimum Value and Minimizer of Table 1

N	Global Minimum Value			Global Minimizer		
	Lagrange interpolation	IYRH's algorithm	Relative error	IYRH's algorithm	Relative error	Lagrange interpolation
1.	1.02e-8	-29763.23334	1.300253442e-9	10	0.0	2.97e-7
2.	1.18e-7	-1.8996	1.379501097e-6	5.14574	8.135723392e-7	4.46e-8
3.	3.15e-8	-12.0312	3.06954672e-6	-6.77458	5.016350666e-7	1.52e-6
	3.15e-8	-12.0312	3.06954672e-6	-0.491391	0.0	1.52e-6
	3.15e-8	-12.0312	3.06954672e-6	5.79179	7,361834923e-7	1.52e-6
4.	1.35e-7	-3.85045	0.0	2.86803	1.110062719e-6	3.69e-7
5.	1.01e-6	-1.48907	0.0	0.966086	2.03449486e-6	2.14e-6
6	6.04e-7	-0.824239	0.0	-0.679579	0.0	1.01e-5
7.	2.89e-6	-1.60131	3.84423173e-6	5.19978	0.0	2.49e-7
8.	6.69e-7	-14.508	0.0	-7.08351	4.948347907e-7	7.16e-7
	6.69e-7	-14.508	0.0	-0.800321	0.0	7.16e-7
	6.69e-7	-14.508	0.0	5.48286	0.0	7.16e-7
9.	3.76e-7	-1.90596	0.0	17.0392	1.108708908e-5	1.10e-5
10	3.08e-7	-7.91673	0.0	7.97867	3.341240937e-6	3.83e-6
11	1.49e-8	-3	0.0	4.76837e-009	0.0	1.66e-7
12	5.74e-9	-1	0.0	3.14159	0.0	3.68e-8
	5.74e-9	-1	0.0	4.71239	1.750581062e-7	3.68e-8
13	2.55e-9	-1.5874	0.0	0.707107	0.0	8.46e-6
14	2.16e-7	-0.788685	0.0	0.22488	4.082015863e-6	3.72e-6
15	6.97e-9	-7.03553	0.0	-0.414214		7.35e-6
16	4.70e-10	0.0111090	0.0	3	0.0	1.80e-9
17	1.68e-9	7	0.0	-3	0.0	1.31e-7
	1.68e-9	7	0.0	3	0.0	1.31e-7
18	1.27e-5	1.97215 e-031	0.0	2	0.0	6.99e-5
19	5.08e-7	0.467511	0.0	0.41032	0.0	6.17e-7
20	1.38e-7	-0.0634905	0.0	-1.19514	0.0	8.53e-6

Table 6. Numerical Results of $f_{67}(x)$ by IYRH's Algorithm

N	k	j/n	x_0	x_k^*	$x_{i,k}^{(j)}$	$x_{M,k}^{(n)}$	$x_{i,k}^{(j+1)}$	$x_{M,k}^{(n)}$	x_k^{Z+}
21	0	1/1	-5	-4.94728	-4.83149	-4.70524	-4.56113		-4.51102
	1	1/1		-4.40585	-4.24739	-4.06127	-3.90347	-3.72308	
		3/2			-3.48405	-3.17118	-2.86885		-1.78366
	2	1/1		-1.34952	-0.762247	-0.434035	-0.211967	-0.0283928	
		3/2			0.194441	0.372753	0.555842	0.720526	
		5/3			0.858072	0.983464	1.13369	1.27036	
		7/4			1.38786	1.49935	1.6191	1.73018	
		9/5			1.84105	1.94761	2.04578	2.13882	
		11/6			2.24384	2.34491	2.43053	2.51298	
		13/7			2.61041	2.70403	2.78338	2.86058	
		15/8			2.94927	3.03458	3.111	3.18578	
		17/9			3.26598	3.34343	3.4181	3.49135	
		19/10			3.56437	3.63526	3.70814	3.77964	
		21/11			3.84724	3.91324	3.98374	4.05282	
		23/12			4.11674	4.17944	4.2469	4.31295	
		25/13			4.37456	4.43522	4.49921	4.56184	
		27/14			4.62208	4.68152	4.74192	4.80109	
		29/15			4.86042	4.91905	4.97605		

Table 7. Comparison of the Numerical Results by IYRH's Algorithm with Two Other Methods

Example	nfI/nfM/nfL	nf*I/nf*M/nf*L	nFI/nFM/NfI	nF*I/nF*M/nF*L
1	27/1361/failed	27/1361/failed	55/1141/failed	62/1211/failed
2	16/170/failed	16/170/failed	16/72/failed	22/132/failed
3	104/339/failed	104/339/failed	35/236/failed	57/296/failed
4	8/418/failed	8/504/failed	9/309/failed	19/429/failed
5	18/505/failed	18/505/failed	20/397/failed	31/457/failed

The meaning of the abbreviations used in Table 8 is as follows:

- nfI, nfM and nfL are the number of function evaluations needed to yield the global minimum of IYRH's, Ma et al.'s and Lucidi and Picialy's algorithms respectively.
- nf*I, nf*M and nf*L are the number of function evaluations needed to satisfy the stopping criterion of IYRH's, Ma et al.'s and Lucidi and Piccially's algorithms respectively.
- nFI , nFM and nFL are the number of filled function evaluations needed to obtain the global minimum of IYRH's, Ma et al.'s and Lucidi and Piccially's algorithms respectively.
- nF*I, nF*M and nF*L are the number of filled function evaluations needed to satisfy the stopping criterion of IYRH's, Ma et al.'s and Lucidi and Piccially's algorithms respectively.
- 5."failed" means the method of Lucidi and Piccially fails to achieve the results.

Table 8. Numerical Results due to 3 Different Initial Points for Example 3 of Table 2

x_0	k	j/n	x_k^*	$x_{i,k}^{(j)}$	$x_{M,k}^{(n)}$	$x_{i,k}^{(j+1)}$	$x_{M,k}^{(n)}$	x_k^{Z+}
-12	0	1/1	-10.8789	-9.99355	-9.10281	-8.00645		-7.87069
	1	1/1	-6.91954	-5.99355	-5.06151	-4.00645		-3.67996
	2	1/1	-2.95985	-1.99355	-1.02048	-0.00645042		0.594662
	3	1/1	1	2.00645	3.02048	3.99355	4.95985	
		3/2		6.00645	7.06151	7.99355	8.91954	
		5/3		10.0065	11.1028	11.9935		
-9.6	0	1/1	-10.8789	-9.99355	-9.10281	-8.00645		-7.87069
	1	1/1	-6.91954	-5.99355	-5.06151	-4.00645		-3.67996
	2	1/1	-2.95985	-1.99355	-1.02048	-0.00645042		0.594662
	3	1/1	1	2.00645	3.02048	3.99355	4.95985	
		3/2		6.00645	7.06151	7.99355	8.91954	
		5/3		10.0065	11.1028	11.9935		
-5.6	0	1/1	-6.91954	-5.99355	-5.06151	-4.00645		-3.67996
	1	1/1	-2.95985	-1.99355	-1.02048	-0.00645042		0.594662
	2	1/1	1	2.00645	3.02048	3.99355	4.95985	
		3/2		6.00645	7.06151	7.99355	8.91954	
		5/3		10.0065	11.1028	11.9935		

8. Comparison and Discussion

The graphical comparison between IYRH's filled function method with other best current filled function methods (1)-(4) included tunneling and bridging methods, will be presented.

8.1. Comparison with the Tunneling Method [8]

The weakness of tunneling method [8] $T(x,\Gamma) = \left(f(x) - f(x_1^*)\right)/[(x - x_1^*)^{\Gamma}(x - x_1^*)]^{\lambda}$ appeared when Newton's method is used since the non-convexity problem. Fortunately, IYRH's filled function can be utilized (Theorem 6 and Theorem 7) using the radius of curvature applied to Newton's method to find the root of non-convex problems. For example 7 [8], the tunneling method can only obtain the global minimizer, whereas IYRH's algorithm can obtain the entire extreme and inflection points in considered domain.

8.2. Comparison with the Bridging Method [41]

The bridging function [41]:

$$f_r = \begin{cases} r(x) & \left((x > x_0)\ or\ \left(f(x) \geq r(x)\right)\right) \\ r(x) - \dfrac{\left(r(x) - f(x)\right)^2}{2\varepsilon} & \left((x > x_0)\ and\ \left(r(x) - \varepsilon < f(x) < r(x)\right)\right) \\ f(x) + \dfrac{\varepsilon}{2} & \left((x > x_0)\ and\ (f(x) < r(x) < \varepsilon)\right) \end{cases}$$

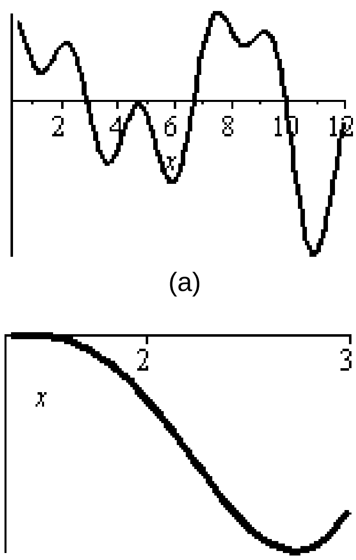
$$f_l = \begin{cases} l(x) & \left((x > x_0)\ or\ \left(f(x) \geq l(x)\right)\right) \\ l(x) - \dfrac{\left(l(x) - f(x)\right)^2}{2\varepsilon} & \left((x > x_0)\ and\ \left(l(x) - \varepsilon < f(x) < l(x)\right)\right) \\ f(x) + \dfrac{\varepsilon}{2} & \left((x > x_0)\ and\ (f(x) < l(x) < \varepsilon)\right) \end{cases}$$

which is strongly depended on the parameters δ_0, δ_1, ε_0, δ and d where δ_0, δ_1, and ε_0 must be predetermined, where $r(x) = r(x, x_0, \delta) = f(x_0) - \delta(x - x_0)$, and $l(x) = l(x, x_0, \delta) = f(x_0) + \delta(x - x_0)$.

The authors suggested that δ_0 should be chosen first and it should big enough so that when $\delta \geq \delta_0$ the computer does not treat $-\delta$ as zero. They suggested$\delta_0 = 10^{-2}\ or\ 10^{-3}$, $\delta_1 = 0.1\ or\ 1$, and d is choosen such that $\delta_1/d^k = \delta_0$. We are lucky since IYRH's algorithm did not face any situation like that.

8.3. Comparison with the Two-Parameter Filled Function

The graph of $f_c(x) = \cos(3x/5)\cos(2x) + \sin(x)\ (0.5 \leq x \leq 12)$ is given in Figure 4 (a) and the graph of $F_{IRH}(x, x_1^*)$ $(x \in [1.34096, 2.73151])$ of f is given in Figure 4 (b) where $x_1^* = 1.34096$. The graphs of the filled functions (1) with $\rho = 1$and $r = 1 - f(x)$, (2) with $\rho = 5$and $\mu = 1$, (3) with $\rho = 1$ and $r = 1 - f(x)$and (1.4) with $x_0 = 2$, $\tau = 10$ are given in Figures 5 (a)-(d) respectively. Contrast to F_{IRH}, (a) F_G has the infinity structure (flat) and strongly depended on ρ and r. Therefore F_G becomes inefficient, (b) F_Z discontinuous at a point $x' \in (2,3)$. This condition makes the minimizing difficult, (c) according to (3), they actually have four parameters to be adjusted (see [29]) and (d) a (see (5)) contain a parameter $\xi > 0$, which is also difficult to be adjusted, and it is clear that F_X is not a one-parameter filled function.



(a)

(b)

Figure 4. Graps of (a) $f(x)$ and (b) F_{IRH} at $x_1^* = 1.34096$

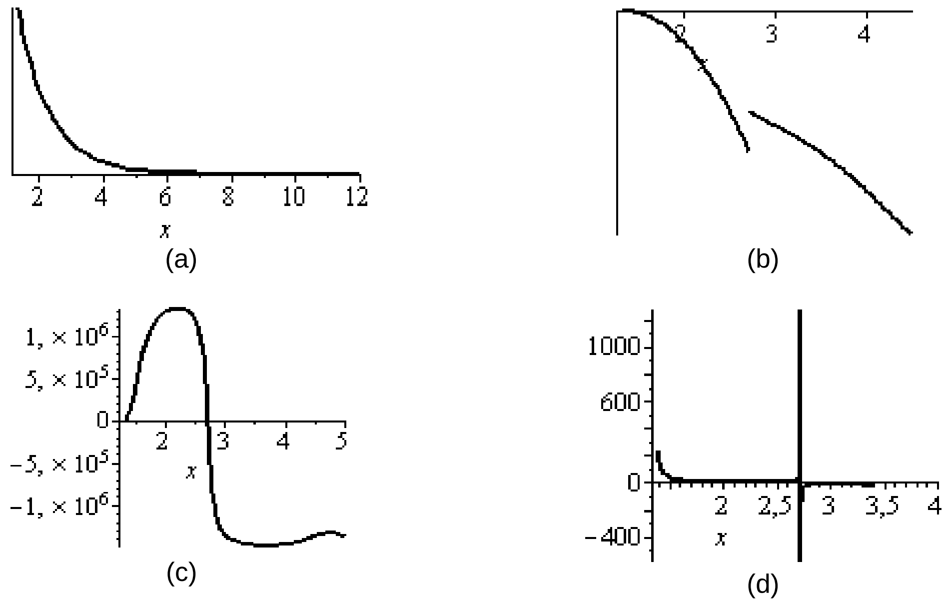


Figure 5. The graphs of (a) F_G, (b) F_Z, (c) F_{LS}, (d) F_X

8.4. Comparison to the Parameter Free Filled Function

Ma et al. [34] suggest a PFFF (6) at x_k^* of f. When applied to Example 9 in Table 1 at $x_k^* = 5.36225$, it yields a graph as in Figure 6. It is clear that, F_M is discontinuous at x_k^{Z+}, and almost flat for x such that $f(x) < f(x_k^*)$ whereas F_{IRH} is continuous as shown in Figure 7.

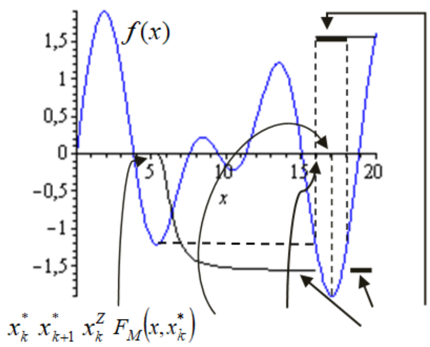


Figure 6. Example 9 in Table 1

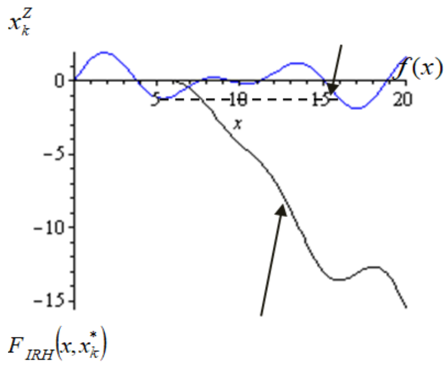


Figure 7. Example 9 in Table 1

8.5. Conclusion

This article introduces a new IYRH's method which absolutely different from other filled function methods, in finding all extreme and inflection points of $f: D \subseteq R \longrightarrow R$ According to the results listed in Tables 4-8, this method never fail compute all those points. Thus, this method is an efficient and reliable method for solving the global optimization problems numerically and analytically. Therefore, the IYRH's method is far more advanced and superior than other most of the filled function methods published in the literature.

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