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Regional gradient optimal control problem governed by a distributed bilinear systems

Maawiya Ould Sidi*, Sid Ahmed Beinane
Mathematics Department, College of Science, Jouf University,
P.O. Box: 2014, Sakaka, Saudi Arabia
*Corresponding author, e-mail: maawiya81@gmail.com, beinane06@gmail.com

Abstract

This paper gives an extension of previous work on gradient optimal control of distributed parabolic systems to the case of distributed bilinear systems which are a type of nonlinear systems. We introduce the notion of flux optimal control of distributed bilinear systems. The idea is trying to achieve a neighborhood of the gradient state of the considered system by minimizing a nonlinear quadratic cost. Using optimization techniques, a method showing how to reach a desired flux at a final time, only on internal subregion of the system domain will be proposed. The proposed simulation illustrates the theoretical approach by commanding the heat bilinear equation flux to a desired profile.

Keywords: *bilinear systems, gradient state, optimal control problem, simulations*

1. Introduction

Infinite dimensional Bilinear systems analysis can formulate many real problems see Lions [1, 2]. The controllability is among the most important analysis notions, within there are many concepts as exact controllability, approximate controllability, regional controllability and so on. In [3], Ball, Marsden, and Slemrod, discussed the controllability for distributed Bilinear systems. Bradly, Lenhart, and Yong, in [4] treated the optimal control of the Velocity term in a Kirchhoff plate equation. A very important applications of optimal control problems, refereeing to the optimal control problem in which the interest state is specified only on ω, a subregion of the system domain. El Jai, Pritchard, Simon, and Zerrik in [5], give an example where the control is required to achieve the temperature at a level in a specified subregion of the furnace.

Many interesting results were developed in the case of parabolic and hyperbolic systems, we cite in particular the result proving that there exist a systems which are not controllable to the whole domain but controllable to a subregion see Zerrik, Kamal [6]. These results were generalized by Zerrik, Kamal in [7] and Zerrik et al in [8] to the case called boundary controllability refereeing to the subregion ω located on the boundary of system domain. The gradient optimal control concerns the optimal control of the gradient state to a subregion of the system domain. The readers can obtain very interest contributions in this field, particularly characterizations of the optimal control that achieves regional gradient controllability by Zerrik et all in [9] and Kamal et al [10] in the case of parabolic linear systems, and Ould Beinane et all [11] in the case of semi linear systems.

For bilinear distributed systems, the notion of regional optimal control is introduced by Zerrik and Ould Sidi in [12-14], showing the existence of an optimal control by a minimizing sequence, solution of the quadratic cost control problem which involves the minimization of the norm control and the final state error and deriving a characterization for optimal control, using the solution of an optimality system. Thereafter, Zerrik. and El Kabouss in [15] gives an extension of previous regional optimal control works to the case of a spatiotemporal damping. El Harraki and Boutoulout in [16] studied the controllability of the wave equation with multiplicative controls. Zine and Ould Sidi in [17, 18] and Zine in [19] treated the regional optimal control problems governed by bilinear hyperbolic distributed systems.

This paper discuss an extension of the previous results on the regional optimal control of distributed systems (linear, semi linear and bilinear) to the case of gradient optimal control of bilinear parabolic system, which constitutes an important progress in system theory. In particular, we treat the problem of regional gradient optimal control of bilinear systems using

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a quadratic nonlinear method. We show the existence of an optimal control solution of the considered problem. Using the optimization techniques, we give a characterization for the optimal control. Numerical simulations are established illustrating successfully the theoretical approach.

2. Preliminary

We consider the following equation, which is governed by a heat bi-linear system

$$\begin{cases} \frac{\partial u}{\partial t} + \Delta u = Q(t)u & \Pi \\ u(x,0) = u_0(x) & \Omega \\ u = 0 & \Sigma \end{cases} \qquad (1)$$

where Ω is an open bounded domain in $IR^n (n = 1,2,3)$, with a regular boundary $\partial\Omega$.

For $T > 0$, $\Pi = \Omega \times]0,T[$, $\Sigma = \partial\Omega \times]0,T[$ and $Q \in L^2(0,T)$ is the control function. The Laplace operator Δ generates the strongly continuous semi-group $(S(t))_{t\geq 0}$ on the state space $L^2(\Omega)$ such that

$$S(t)u_Q(t) \quad = \sum_{n=1}^{+\infty} e^{\lambda_n t} < u_Q(t), \phi_n > \phi_n. \qquad (2)$$

where λ_n is the eigenvalues of Δ and ϕ_n its associate eigenfunctions. For a given $u_0 \in H^1(\Omega)$, the system (1) may be written as:

$$u(t) = S(t)u_0 + \int_0^t S(t-s)Q(s)u(s)ds. \qquad (3)$$

and solutions of (3) are often called mild solutions of (1). The existence of a unique solution $u_Q(x,t)$ in $L^2(0,T; H_0^1(\Omega))$ satisfying (3) follows from standard results in [20, 21]. For $\omega \in \Omega$, we define the restriction operator to ω by:

$$\begin{array}{llll} \chi_\omega : (L^2(\Omega))^n & \longrightarrow & (L^2(\omega))^n \\ \qquad u & \longrightarrow & \chi_\omega u = u|_\omega \end{array}$$

and χ_ω^* its adjoint given by

$$\chi_\omega^* u = \begin{cases} u \ \ in\Omega \\ 0 \in \Omega\backslash\omega \end{cases}$$

and

$$\begin{array}{lll} \tilde{\chi}_\omega : (L^2(\Omega)) & \longrightarrow & (L^2(\omega)) \\ \qquad u & \longrightarrow & \tilde{\chi}_\omega u = u|_\omega \end{array}$$

let ∇ the operator defined by

$$\begin{array}{lll} \nabla : H^1(\Omega) & \longrightarrow & (L^2(\Omega))^n \\ \qquad u & \longrightarrow & \nabla u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}) \end{array}$$

with adjoint ∇^*.

2.1. Definition

The system (1) is said to be weakly regionally gradient controllable on $\omega \subset \Omega$ if for all $g^d \in (L^2(\omega))^n$ and $\varepsilon > 0$ there exists a control $Q \in L^2[0,T]$ such that

$$||\chi_\omega \nabla u_Q(T) - g^d||_{(L^2(\omega))^n} \leq \varepsilon$$

where $g^d = (y_1^d, \ldots. y_n^d)$ is the gradient of the desired state in the space $L^2(\omega)$.

Our main objective is to solve the regional gradient quadratic control problem governed by the bi-linear distributed in (1)

$$\min_{Q \in L^2([0,T])} J_\varepsilon(Q). \tag{4}$$

where the gradient quadratic cost J_ε is defined for $\varepsilon > 0$ by

$$\begin{aligned} J_\varepsilon(Q) &= \frac{1}{2} \| \chi_\omega \nabla u(T) - g^d \|^2_{(L^2(\omega))^n} + \varepsilon \| Q(t) \|^2_{L^2([0,T])} \\ &= \frac{1}{2}\sum_{i=1}^n \| \tilde{\chi}_\omega \frac{\partial u(T)}{\partial x_i} - y_i^d \|^2_{L^2(\omega)} + \varepsilon \| Q(t) \|^2_{L^2([0,T])} \end{aligned} \tag{5}$$

Quadratic optimal control problem governed by bilinear systems aims in general to steer the state of a considered system to a desired profile. Many references use quadratic cost us (5), we cite for example Addou and Benbrik in [22], Bradly and Lenhart in [23], Bradly et all in [4], and Lenhart [24]. In applications there are many motivations of the regional gradient optimal control problems governed by bilinear systems, for example in thermal isolation problems it happens that the control is maintained to reducing the gradient temperature before the brick see [6-8]. The original goal of this paper is to street the gradient state of the bilinear system (1) to the desired state $g^d(x)$ by minimizing objective functional (5), and characterize an optimal control $Q^* \in L^2(0,T)$ such that $J_\varepsilon(Q^*) = \min_{Q \in L^2(0,T)} J_\varepsilon(Q)$.

3. Existence of Solution

Firstly, we prove our main theorem of this section.

3.1. Theorem

There exists a pair $(\bar{u}, Q^*) \in C([0,T]; H_0^1(\Omega)) \times L^2([0,T])$, such that $\bar{u}$ is the unique solution of

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = -\Delta u(x,t) + Q^*(t)u(x,t) & \Pi \\ u(x,0) = u_0(x) & \Omega \end{cases} \tag{6}$$

and Q^* is solution of the problem (4).

Proof. The set $\{J_\varepsilon(Q) | \ Q \in L^2([0,T])\}$ is a nonempty set of IR^+, then it admit a lower bounded. We choose $(Q_n)_n$ a minimizing sequence such that

$$J^* = \lim_{n\to+\infty} J(Q_n) = \inf_{Q\in L^2([0,T])} J_\varepsilon(Q)$$

$J_\varepsilon(Q_n)$ is then bounded, it follows that $||Q_n||_{L^2([0,T])} \leq M$, for a positive constant M. Using $u_Q(x,t)$ the weak solution of (1) in $W = L^2(0,T; H_0^1(\Omega))$, and from (3), we deduce that

$$||u_Q(t)||_W \leq \left(||u_0||_{L^2(\Omega)} + \int_0^t |Q(s)|||u_Q(s)||_W ds\right)$$

by Gronwall inequality, we have

$$||u_Q(t)||_W \leq C\exp(MT) \tag{7}$$

where $C = ||u_0||_{L^2(\Omega)}$, and from there, we deduce that $u_n(x,t) = u_{Q_n}(x,t)$ is bounded. From the bounds of Q_n and $u_n(x,t)$ follows

$$||\Delta(y_n)||_W \ \leq \ M_1, ||Q_n(u_n)||_W \ \leq \ M_2, \ and \ \ ||u_{n'}||_W \ \leq \ M_3$$

where M_1, M_2 and M_3 are three positive constants. From the priori estimates, we can extract a subsequences such as:

$$\begin{array}{lll} Q_n \rightharpoonup Q^* & weakly\ in & L^2(0,T) \\ u_n \rightharpoonup \bar{u} & weakly\ in & W \\ \Delta u_n \rightharpoonup \chi & weakly\ in & W \\ u_n(Q_n) \rightharpoonup \Lambda & weakly\ in & W \\ u_{n'} \rightharpoonup \Psi & weakly\ in & W \end{array} \tag{8}$$

by classical argument, we check that $\bar{u}(0) = u_0$, then by limit as $n \longrightarrow \infty$ the system (6) gives $\bar{u}' = \Psi$, $\Delta\bar{u} = \chi$ and $Q^*\bar{u} = \Lambda$. Furthermore $\bar{u} = u(Q^*)$. To prove that Q^* is optimal, we use the lower semi continuity of $J_\varepsilon(Q)$, and applying Fatou's Lemma we have

$$\begin{array}{lll} J(Q^*) & = & \frac{1}{2}\inf_n \sum_{i=1}^n \int_\omega (\tilde{\chi}_\omega \frac{\partial u_n}{\partial x_i} - y_i^d)^2 dx + \varepsilon \int_0^T Q_n^2(t)dt \\ & \leq & \lim_{n\longrightarrow\infty} J_\varepsilon(Q_n) = \inf_Q J_\varepsilon(Q) \end{array} \tag{9}$$

which prove that Q^* is optimal for the problem (4).

4. Characterization of Solution

To formulate an explicit solution of the optimal problem (4), we propose an adjoint equation by differentiating the quadratic cost $J_\varepsilon(Q)$ respecting to Q. Next Lemma study the differential of $Q \longrightarrow u(Q)$ with respect to Q.

4.1. Lemma

The function

$$\begin{array}{ccc} L^2(0,T) & \longrightarrow & C([0,T]; H^1(\Omega)), \\ Q & \longrightarrow & u(Q) \end{array}$$

solution of (6) is differentiable and its differential ψ verify the system

$$\begin{cases} \frac{\partial\psi(x,t)}{\partial t} = -\Delta\psi(x,t) + Q^*(t)\psi(x,t) + h(t)\bar{u}(x,t) & \Pi \\ \psi(x,0) = \psi_0(x) = 0 & \Omega \end{cases} \tag{10}$$

with $\bar{u} = u(Q^*)$, $h \in L^2([0,T])$, and $d(u(Q^*))h$ is the differential of $Q \to u(Q)$ respecting Q^*.
Proof. Since ψ is solution of the (10), we have

$$||\psi||_W \leq k_1 ||\bar{u}||_{L^\infty(0,T;H_0^1(\Omega))} ||h||_{L^2([0,T])}$$

and

$$||\psi'||_W \leq k_2 ||\bar{u}||_{L^\infty(0,T;H_0^1(\Omega))} ||h||_{L^2([0,T])}$$

consequently,

$$||\psi||_{C([0,T];H_0^1(\Omega))} \leq k_3 ||h||_{L^2([0,T])}$$

we deduce that $h \in L^2([0,T]) \to \psi \in C((0,T); H_0^1(\Omega))$ is bounded, see [13]. Put $u_h = u(Q^* + h)$ and $\varphi = u_h - \bar{u}$, then φ verify

$$\begin{cases} \frac{\partial\varphi(x,t)}{\partial t} = -\Delta\varphi(x,t) + Q^*(t)\varphi(x,t) + h(t)u_h(x,t) & \Pi \\ \varphi(x,0) = 0 & \Omega \end{cases} \tag{11}$$

consequently

$$||\varphi||_{L^\infty([0,T];H_0^1(\Omega))} \leq k_4||h\,\|_{L^2([0,T])}$$

where $k_i, \{i = 1,2,3,4\}$, and k are positive constants. Let the map $\phi = \varphi - \psi$ which is solution of

$$\begin{cases} \frac{\partial\phi(x,t)}{\partial t} = -\Delta\phi(x,t) + Q^*(t)\phi(x,t) + h(t)\varphi(x,t) & \Pi \\ \phi(x,0) = 0 & \Omega \end{cases} \quad (12)$$

$\phi \in C([0,T]; H_0^1(\Omega))$, and we have

$$||\phi||_{C([0,T];L_0^1(\Omega))} \leq k||h||^2_{L^2([0,T])}$$

furthermore

$$||u(Q^* + h) - u(Q^*) - d(u(Q^*))h||_{C(0,T;H_0^1(\Omega))} \leq k||h||^2_{L^2([0,T])}.$$

Next, we consider the family of optimality systems

$$\begin{cases} \frac{\partial p_i(x,t)}{\partial t} = \Delta p_i(x,t) - Q_\varepsilon^*(t)p_i(x,t) & Q \\ p_i(x,T) = (\frac{\partial u(T)}{\partial x_i} - \tilde{\chi}_\omega^* y_i^d) & \Omega \end{cases} \quad (13)$$

where $\tilde{\chi}_\omega^*$ is the adjoint of $\tilde{\chi}_\omega$ defined from $L^2(\omega) \longrightarrow L^2(\Omega)$ by

$$\tilde{\chi}_\omega^* u(x) = \begin{cases} u(x) & x \in \omega \\ 0 & x \in \Omega\backslash\omega \end{cases}$$

the following lemma gives the differential of $J_\varepsilon(Q)$, respecting to Q.

4.2. Lemma

If $Q_\varepsilon \in L^2(0,T)$ the optimal control solution of (4), ψ is the solution of (10) and p_i is the solution of (13), then

$$\begin{aligned} \lim_{\beta\longrightarrow 0} \frac{J_\varepsilon(Q_\varepsilon+\beta h)-J_\varepsilon(Q_\varepsilon)}{\beta} &= \sum_{i=1}^n \int_\omega \tilde{\chi}_\omega^*\tilde{\chi}_\omega \left[\int_0^T \frac{\partial p_i}{\partial t}\frac{\partial\psi(x,t)}{\partial x_i}dt + \int_0^T p_i \frac{\partial}{\partial x_i}(\frac{\partial\psi}{\partial t})dt\right]dx \\ &+ \int_0^T 2\varepsilon h Q_\varepsilon dt. \end{aligned} \quad (14)$$

proof. The quadratic cost $J_\varepsilon(Q_\varepsilon)$ defined by (5), can be write in the following form

$$J_\varepsilon(Q_\varepsilon) = \frac{1}{2}\sum_{i=1}^n \int_\omega (\tilde{\chi}_\omega \frac{\partial u}{\partial x_i} - y_i^d)^2 dx + \varepsilon\int_0^T Q_\varepsilon^2(t)dt \quad (15)$$

let $u_\beta = u(Q_\varepsilon + \beta h)$ and $u = u(Q_\varepsilon)$, using (15) we have

$$\begin{aligned} \lim_{\beta\longrightarrow 0} \frac{J_\varepsilon(Q_\varepsilon+\beta h)-J_\varepsilon(Q_\varepsilon)}{\beta} &= \lim_{\beta\longrightarrow 0} \sum_{i=1}^n \frac{1}{2}\int_\omega \frac{(\tilde{\chi}_\omega\frac{\partial u_\beta}{\partial x_i}-y_i^d)^2-(\tilde{\chi}_\omega\frac{\partial u}{\partial x_i}-y_i^d)^2}{\beta}dx \\ &+ \lim_{\beta\longrightarrow 0} \varepsilon\int_0^T \frac{(Q_\varepsilon+\beta h)^2-Q_\varepsilon^2}{\beta}(t)dt. \end{aligned} \quad (16)$$

consequently

$$\lim_{\beta\to 0}\frac{J_\varepsilon(Q_\varepsilon+\beta h)-J_\varepsilon(Q_\varepsilon)}{\beta}$$

$$\begin{aligned}
&= \lim_{\beta\to 0}\sum_{i=1}^{n}\frac{1}{2}\int_\omega \tilde{\chi}_\omega \frac{(\frac{\partial u_\beta}{\partial x_i}-\frac{\partial u}{\partial x_i})}{\beta}(\tilde{\chi}_\omega\frac{\partial u_\beta}{\partial x_i}+\tilde{\chi}_\omega\frac{\partial u}{\partial x_i}-2y_i^d)dx \\
&+ \int_0^T (2\varepsilon h Q_\varepsilon+\beta\varepsilon h^2)dt \\
&= \sum_{i=1}^{n}\int_\omega \tilde{\chi}_\omega \frac{\partial\psi(x,T)}{\partial x_i}\tilde{\chi}_\omega(\frac{\partial u(x,T)}{\partial x_i}-\tilde{\chi}_\omega^* y_i^d)dx+\int_0^T 2\varepsilon h Q_\varepsilon dt \\
&= \sum_{i=1}^{n}\int_\omega \tilde{\chi}_\omega \frac{\partial\psi(x,T)}{\partial x_i}\tilde{\chi}_\omega p_i(x,T)dx+2\varepsilon\int_0^T hQ_\varepsilon dt
\end{aligned} \tag{17}$$

from (13) and (17), we deduce that

$$\begin{aligned}
\lim_{\beta\to 0}\frac{J_\varepsilon(Q_\varepsilon+\beta h)-J_\varepsilon(Q_\varepsilon)}{\beta} &= \sum_{i=1}^{n}\int_\omega \tilde{\chi}_\omega^*\tilde{\chi}_\omega\left[\int_0^T \frac{\partial p_i}{\partial t}\frac{\partial\psi(x,t)}{\partial x_i}dt+\int_0^T p_i\frac{\partial}{\partial x_i}(\frac{\partial\psi}{\partial t})dt\right]dx \\
&+ \int_0^T 2\varepsilon h Q_\varepsilon dt.
\end{aligned} \tag{18}$$

which finishes the proof of this Lemma. Now, we are ready to characterize the optimal control solution of (5), using our defined family of optimality systems.

4.3. Theorem

If $Q_\varepsilon \in L^2(0,T)$ is an optimal control, and $u_\varepsilon = u(Q_\varepsilon)$ its associate state solution of the system (1), then

$$Q_\varepsilon(t) = \frac{-1}{2\varepsilon}\sum_{i=1}^{n}\langle\tilde{\chi}_\omega\frac{\partial u(x,t)}{\partial x_i};\tilde{\chi}_\omega p_i(t)\rangle_{L^2(\omega)} \tag{19}$$

is solution of the problem (??), where $p_i \in C([0,T];H_0^1(\Omega))$ is the unique solution of the adjoint system (13). Proof. Let $h \in L^\infty(0,T)$ such that $Q_\varepsilon+\beta h \in L^2(0,T)$ for $\beta > 0$. The minimum of J_ε is achieved at Q_ε, then

$$0 \le \lim_{\beta\to 0}\frac{J_\varepsilon(Q_\varepsilon+\beta h)-J_\varepsilon(Q_\varepsilon)}{\beta}. \tag{20}$$

using Lemma (4.2) replacing $\frac{\partial\psi}{\partial t}$ in the system (10), we have

$$\begin{aligned}
0 &\le \lim_{\beta\to 0}\frac{Q_\varepsilon(u_\varepsilon+\beta h)-Q_\varepsilon(u_\varepsilon)}{\beta} \\
&= \sum_{i=1}^{n}\int_\omega \tilde{\chi}_\omega^*\tilde{\chi}_\omega\left[\int_0^T \frac{\partial\psi}{\partial x_i}\frac{\partial p_i}{\partial t}dt+\int_0^T(-\Delta\frac{\partial\psi}{\partial x_i}+Q_\varepsilon(t)\frac{\partial\psi}{\partial x_i}+h(t)\frac{\partial u}{\partial x_i})p_i dt\right]dx+\int_0^T 2\varepsilon h Q_\varepsilon dt.
\end{aligned} \tag{21}$$

and from the system (13) we have

$$\begin{aligned}
0 &\le \sum_{i=1}^{n}\int_\omega \tilde{\chi}_\omega^*\tilde{\chi}_\omega\left[\int_0^T \frac{\partial\psi}{\partial x_i}(\frac{\partial p_i}{\partial t}+\Delta p_i+Q(t)p_i)+h(t)\frac{\partial u}{\partial x_i}p_i dt\right]dx+\int_0^T 2\varepsilon h Q_\varepsilon dt \\
&= \int_0^T 2\varepsilon h Q_\varepsilon dt+\sum_{i=1}^{n}\int_0^T h(t)\langle\tilde{\chi}_\omega\frac{\partial u}{\partial x_i};\tilde{\chi}_\omega p_i\rangle_{L^2(\omega)} \\
&= \int_0^T h(t)\left[2\varepsilon Q_\varepsilon(t)+\sum_{i=1}^{n}\langle\tilde{\chi}_\omega\frac{\partial u}{\partial x_i};\tilde{\chi}_\omega p_i(t)\rangle_{L^2(\omega)}\right]dt.
\end{aligned} \tag{22}$$

note that h = h(t) is an arbitrary function with $Q_\varepsilon+\beta h \in L^2(0,T)$ for all small β, by a standard control argument involving the sign of the variation h depending on the size of Q_ε, we obtain the desired characterization of Q_ε, namely,

$$Q_\varepsilon(t) = \frac{-1}{2\varepsilon}\sum_{i=1}^{n} \langle \tilde{\chi}_\omega \frac{\partial u(x,t)}{\partial x_i}; \tilde{\chi}_\omega p_i(t)\rangle_{L^2(\omega)} \quad (23)$$

the existence of a solution to the adjoint (13) is similar to existence of solution to the state equation since

$$(\frac{\partial u(T)}{\partial x_i} - \tilde{\chi}_\omega^* y_i^d) \text{ in } C([0,T], H_0^1(\Omega))$$

4.4. Remarks

In the case of Neumann boundary conditions, all contributions can be easily generalized. The map $u \longrightarrow Q(t)u$ is not use as a special case. The same results hold with other types of damping.

5. Simulations

For simulations, we Choose the one dimensional bi-linear equation

$$\begin{cases} \frac{\partial u}{\partial t} + \alpha \frac{\partial^2 u}{\partial x^2} = \beta Q(t)u & [0,1] \\ u(x,0) = u_0(x) = x^2, & [0,1] \\ u = 0 & at \;\; x = 0,1 \end{cases} \quad (24)$$

the operator $-\alpha \frac{\partial^2}{\partial x^2}$ admits a set of eigenfunctions $\phi_n(.)$ associated to the eigenvalues λ_n given by:

$$\phi_n(x) = \sqrt{2}\sin(n\pi x); \lambda_n = \alpha n^2\pi^2, \; n \geq 1.$$

while the operator of the system (24) and the perturbation $Q(t)u$ commute, the solution of (24) can be write as

$$u_m(t) \;\; = \sum_{n=1}^{n=M} e^{\alpha n^2\pi^2 t} < e^{\beta \int_0^t Q_m(s)ds} x^2, \sqrt{2}\sin(n\pi x) > \sqrt{2}\sin(n\pi x). \quad (25)$$

and its gradient

$$\frac{\partial u_m(t)}{\partial x} \;\; = \sum_{n=1}^{n=M} e^{\alpha n^2\pi^2 t} < e^{\beta \int_0^t Q_m(s)ds} x^2, \sqrt{2}\sin(n\pi x) > \sqrt{2}n\pi\cos(n\pi x). \quad (26)$$

where the optimal control Q_m is calculated by choosing $\varepsilon = \frac{1}{m}$ and

$$\begin{cases} Q_{m+1}(t) = \frac{-m}{2}\langle \tilde{\chi}_\omega \frac{\partial u_m(x,t)}{\partial x}; \tilde{\chi}_\omega p_m(t)\rangle_{L^2(\omega)} \\ Q_0 = 0 \end{cases} \quad (27)$$

and p is the solution of

$$\begin{cases} \frac{\partial p_m(x,t)}{\partial t} = \alpha \frac{\partial^2 p_m(x,t)}{\partial x^2} - \beta Q_m(t) p_m(x,t) & [0,1] \\ \\ p_m(x,T) = (\frac{\partial u_Q(T)}{\partial x} - \tilde{\chi}_\omega^* g^d(x)) & [0,1] \end{cases} \quad (28)$$

the formula (25) is the mild solution of the system (24) calculate using the semi group associated to the operator $-\alpha \frac{\partial^2}{\partial x^2}$ and the formula (26) its derivative, see [21]. The solution of the (28) with final condition is

$$p_m(t) \;\; = \sum_{n=1}^{n=M} e^{\alpha n^2\pi^2 (T-t)} < e^{\beta \int_{T-t}^{t} Q_m(T-s)ds} (\frac{\partial u_Q(T)}{\partial x} - \tilde{\chi}_\omega^* g^d(x)), \sqrt{2}\sin(n\pi x) > \sqrt{2}\sin(n\pi x) \quad (29)$$

the formula (27) is the minimizing bounded sequence of optimal control deduced from the theorem 4.3. It admits a subsequence convergent, which allow us to consider the following convergent algorithm for numerical implementation of the above results. The Algorithim show in Table 1.

Table 1. Algorithm

Step 1	Step 2	Step 3
Defined the initial data of the problem The control time T The gradient state g^d The error ε The subregionω	Until $\|Q_{(m+1)}$-$Q_m\|$≤ε repeat Compute $\frac{\partial u_m}{\partial t}(T)$ by (26) Compute $p_m(t)$ by the formula (28) Compute Q_{m+1} by the formula (27)	Q_m such that $\| Q_{m+1} - Q_m \| \leq \varepsilon$ is the solution of (4)

5.1. Remarks

a. Let consider the error $E = \| \frac{\partial u_Q(T)}{\partial t} - g^d \|^2_{L^2(\omega)}$. It is depending of the subregion ω and of the amplitude of the desired state chosen as shown bellow.
b. The truncation M defined in (26) will be such that $E \leq \varepsilon$, simulations in general, show that a big choice of M is not preferred due to number of iterations and accumulation of errors.
c. All simulations are obtained by using complex numerical program en FORTRAN 95.
d. Optimal control problems using optimization methods as in [25] are still under considerations.

5.2. Example

We choose $\Omega =]0,1[$, $T = 2$, $\alpha = \beta = 0.01$ and applying the previous algorithm, we propose two examples of simulations. We choose the desired states is g^d (x)=x(1-x)(x+1) chosen for numerical considerations and ω=]0.5,0.7[.Figure 1 shows how the reached position is very close to the desired position on ω, the desired state is obtained with error $E = 2.01 \times 10^{-4}$.

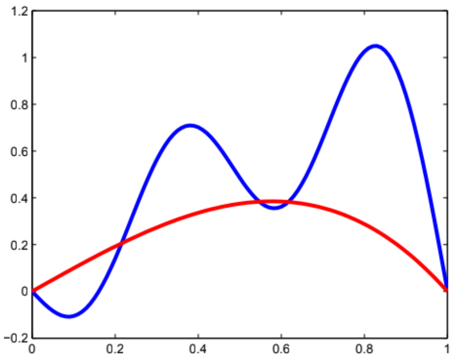

Figure 1. Desired (red line) and final (blue line) gradient position on ω

6. Relations Subregion-Error and Amplitude-Error

Numerically, we show how the error grow with respect to the subregion Table 2 and with respect to the amplitude of the desired state Table 3.

Table 2. Relation Subregion-Error

Subregionω	Error E
]0.4, 0.6 [	$2.01 * 10^{-4}$
] 0.38, 0.7 [	$3.07 * 10^{-4}$
] 0.25, 0.75 [	$1.03 * 10^{-3}$
] 0.2, 0.81 [	$2.11 * 10^{-2}$
] 0.03, 0.88[	$5.1 * 10^{-2}$

Table 3. Relation Amplitude-Error

Amplitude	Error E
0.4	$2.01 * 10^{-4}$
0.45	$3.07 * 10^{-4}$
0.6	$2.01 * 10^{-3}$
0.7	$3.4 * 10^{-2}$
0.9	$1.01 * 10^{-2}$

7. Conclusion

This paper considers for the first time the problem of regional gradient optimal control of infinite dimensional bilinear systems. We have shown the existence of solution of such problem and we have proposed a characterization of its solution. The results have been tested successfully through numerical simulations.

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