

# A modification of fuzzy arithmetic operators for solving near-zero fully fuzzy matrix equation

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## ABSTRACT

Matrix equations have its own important in the field of control system engineering particularly in the stability analysis of linear control systems and the reduction of nonlinear control system models. There are certain conditions where the classical matrix equation are not well equipped to handle the uncertainty problems such as during the process of stability analysis and reduction in control system engineering. In this study, an algorithm is developed for solving fully fuzzy matrix equation particularly for  $\tilde{A}\tilde{X}\tilde{B}-\tilde{X}=\tilde{C}$ , where the coefficients of the equation are in near-zero fuzzy numbers. By modifying the existing fuzzy multiplication arithmetic operators, the proposed algorithm exceeds the positive restriction to allow the near-zero fuzzy numbers as the coefficients. Besides that, a new fuzzy subtraction arithmetic operator has also been proposed as the existing operator failed to satisfy the both sides of the near-zero fully fuzzy matrix equation. Subsequently, Kronecker product and *Vec*-operator are adapted with the modified fuzzy arithmetic operator in order to transform the fully fuzzy matrix equation to a fully fuzzy linear system. On top of that, a new associated linear system is developed to obtain the final solution. A numerical example and the verification of the solution are presented to demonstrate the proposed algorithm.

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## 1. INTRODUCTION

There are many types of matrix equations that have been modelled in various applications [1] particularly in control system engineering [2, 3]. Basically, control system engineering is used to design the feedback loops system [4]. The example applications that related to the feedback loops systems are medical imaging acquisition system [5], image restoration [6], model reduction [7], signal processing [8] and stochastic control [9]. According to [10], matrix equation plays the role as an equation solver for the control system model. In dealing with any real applications, it is possible that any uncertainty conditions could occur, for example, if there exist any conflicting requirements and instability of the environmental conditions during the system process. If there is any existence of noise or unnecessary elements during the process, it would also

distract the system [11]. In this case, the existing matrix equations sometimes are not well equipped to handle those conditions. Therefore, one of the approaches that can be taken is to adapt the fuzzy numbers as the coefficients of the matrix equation [12].

In the past few years, many researchers proposed their algorithms in solving matrix equations with parameters in fuzzy numbers. This equation is known as the fully fuzzy matrix equation (FFME). Otadi and Mosleh [13] are the pioneers in this field, who has applied linear programming technique to obtain a positive solution for arbitrary FFME,  $\tilde{A}\tilde{X}_m = \tilde{B}_m$ . Apart from that, there is a study which has extended the algorithm used in solving the fully fuzzy linear system (FFLS) to solve the FFME  $\tilde{A}\tilde{X}\tilde{B} = \tilde{C}$  [14]. Subsequently, in 2015, Shang et al. [15] proposed their algorithm in solving fully fuzzy Sylvester matrix equation (FFSE),  $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$  by applying the arithmetic multiplication operator, which has been previously proposed in Dehghan et al. [16]. On the other hand, Malkawi et al. [17] have proposed an algorithm which offers faster computational compared to Shang et al. [15]. While in 2020, Elsayed et al. [18] carried out a study in solving the FFME of  $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ , which considering the entries of the equation are in trapezoidal fuzzy numbers.

In this paper, we are propose an algorithm to solve the FFME of

$$\tilde{A}\tilde{X}\tilde{B} - \tilde{X} = \tilde{C} \quad (1)$$

considering the fuzzy coefficient  $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$  or  $\tilde{B} = (\tilde{b}_{ij})_{n \times n}$  is a near-zero fuzzy number, while  $\tilde{C} = (\tilde{c}_{ij})_{m \times n}$  is an arbitrary fuzzy coefficient and  $\tilde{X} = (\tilde{x}_{ij})_{m \times n}$  is the solution of the FFME. This equation has been previously solved by Daud et al. [19] in 2018. Unfortunately the algorithm proposed is only limited to non-singular and positive fuzzy matrices. This limitation has motivated us to construct an algorithm to solve the (1) without any restrictions. Moreover, in real-life applications, the coefficients of the FFME can either be positive, negative or near-zero fuzzy numbers.

In developing the algorithm, the existing fuzzy multiplication arithmetic operators are modified as the existing operators introduced by [17] and [20] are not applicable to perform the multiplication involving near-zero fuzzy numbers. Besides that, a new fuzzy subtraction operation is also developed in solving the FFME, since the existing operator is inadequate to subtract a near-zero fuzzy number to a positive fuzzy number. Subsequently, the modified fuzzy arithmetic operator is adapted with the Kronecker product and *Vec*-operator in converting the FFME to a simpler form of equation, which is a fully fuzzy linear system (FFLS). Later on, the solution is obtained by means of associated linear system (ALS) which has been established based on the modified fuzzy multiplication arithmetic operator.

The remaining part of the paper proceeds as follows. In Section 2, some preliminaries on the fuzzy numbers and Kronecker product are shown. Then in Section 3, the theoretical foundation which supports the developed algorithm are established. In Section 4, the developed algorithm for solving the FFME of (1) is shown. Moving on, a numerical example and verification of the solution are illustrated in Section 5. Finally, the conclusion is drawn in Section 6.

## 2. PRELIMINARIES

### 2.1. Fundamental concepts of matrix and set theory

The fundamental concept of matrix theory is important in order to solve the matrix equations. Some fundamentals of matrix theory are defined in the following:

**Definition 1.** [21] Let  $N$  be a  $3 \times 3$  block matrix, such that

$$N = \left( \begin{array}{cc|c} A & B & C \\ D & E & F \\ \hline G & H & I \end{array} \right), \quad (2)$$

then,

$$\begin{aligned} |N| &= \det \left[ \begin{pmatrix} A & B \\ D & E \end{pmatrix} - \begin{pmatrix} C \\ F \end{pmatrix} I^{-1} (G \ H) \right] \times \det[I] \\ &= \det \begin{bmatrix} A - CI^{-1}G & B - CI^{-1}H \\ D - FI^{-1}G & E - FI^{-1}H \end{bmatrix} \times \det[I] \end{aligned} \quad (3)$$

**Remark 1.** For the block matrix  $3 \times 3$  such that

$$P = \left( \begin{array}{c|cc} A & B & C \\ \hline D & E & F \\ G & H & I \end{array} \right). \quad (4)$$

Clearly that based on Definition 1, the determinant  $P$  is given as follows:

$$\begin{aligned} |P| &= \det \left[ \begin{pmatrix} E & F \\ H & I \end{pmatrix} - \begin{pmatrix} D \\ G \end{pmatrix} A^{-1} (B \ C) \right] \times \det[A] \\ &= \det \begin{bmatrix} E - DA^{-1}B & F - DA^{-1}C \\ H - GA^{-1}B & I - GA^{-1}C \end{bmatrix} \times \det[A] \end{aligned} \quad (5)$$

**Definition 2.** [22] Let  $A$  and  $B$  be sets. The union of  $A$  and  $B$  is the set of  $A \cup B = \{x : x \in A \text{ or } x \in B\}$ .

## 2.2. Theory of fuzzy numbers

The following definition describing the theory of fuzzy numbers has been introduced since 1965 by Zadeh [23].

**Definition 3.** Let  $X$  be a nonempty set. A fuzzy set  $\tilde{A}$  in  $X$  is characterized by its membership function

$$\mu_{\tilde{A}} : X \rightarrow [0, 1] \quad (6)$$

and  $\mu_{\tilde{A}}(x)$  represents the degree of membership of element  $x$  in fuzzy set  $\tilde{A}$  for each  $x \in X$ .

In this study, the representation of fuzzy numbers is based on the triangular fuzzy numbers.

### 2.2.1. Triangular fuzzy number

**Definition 4.** A fuzzy number  $\tilde{M} = (m, \alpha, \beta)$  is said to be a triangular fuzzy number (TFN), if its membership function is given by:

$$\mu_{\tilde{M}}(x) = \begin{cases} 1 - \frac{m-x}{\alpha}, & m - \alpha \leq x \leq m, \alpha > 0, \\ 1 - \frac{x-m}{\beta}, & m \leq x \leq m + \beta, \beta > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (7)$$

In this case,  $m$  is the mean value of  $\tilde{M}$ , whereas  $\alpha$  and  $\beta$  are the right and left spreads, respectively.

**Definition 5.** A fuzzy number  $\tilde{M} = (m, \alpha, \beta)$  is called as an arbitrary fuzzy number where it may be positive, negative or near zero which can be classified as follows:

- $\tilde{M}$  is a positive(negative) fuzzy number iff  $m - \alpha \geq 0$  ( $\beta + m \leq 0$ ).
- $\tilde{M}$  is a zero fuzzy number if  $(m = 0, \alpha, \beta = 0)$ .
- $\tilde{M}$  is a near zero fuzzy number iff  $m - \alpha \leq 0 \leq \beta + m$ .

The following definitions describe some important arithmetic operations of TFN [20].

**Definition 6.** The arithmetic operations of two TFN,  $\tilde{M} = (m, \alpha, \beta)$  and  $\tilde{N} = (n, \gamma, \delta)$ , are as follows:

i. Addition:

$$\tilde{M} \oplus \tilde{N} = (m, \alpha, \beta) \oplus (n, \gamma, \delta) = (m + n, \alpha + \gamma, \beta + \delta). \quad (8)$$

ii. Opposite:

$$-\tilde{M} = -(m, \alpha, \beta) = (-m, \beta, \alpha). \quad (9)$$

iii. Subtraction:

$$\begin{aligned} (m, \alpha, \beta) \ominus (n, \gamma, \delta) &= (m, \alpha, \beta) \oplus -(n, \gamma, \delta) \\ &= (m, \alpha, \beta) \oplus (-n, \delta, \gamma) \\ &= (m - n, \alpha + \delta, \beta + \gamma). \end{aligned} \quad (10)$$

iv. *Multiplication:*

- If  $\tilde{M} > 0$  and  $\tilde{N} > 0$ , then

$$\tilde{M} \otimes \tilde{N} = (m, \alpha, \beta) \otimes (n, \gamma, \delta) \cong (mn, m\gamma + n\alpha, m\delta + n\beta) \quad (11)$$

- If  $\tilde{M} < 0$  and  $\tilde{N} > 0$ , then

$$\tilde{M} \otimes \tilde{N} = (m, \alpha, \beta) \otimes (n, \gamma, \delta) \cong (mn, n\alpha - m\delta, n\beta - m\gamma) \quad (12)$$

- If  $\tilde{M} < 0$  and  $\tilde{N} < 0$ , then

$$\tilde{M} \otimes \tilde{N} = (m, \alpha, \beta) \otimes (n, \gamma, \delta) \cong (mn, -n\beta - m\delta, -n\alpha - m\gamma) \quad (13)$$

Based on the multiplication arithmetic operator in (11) to (13), there is no operator applicable for a near-zero fuzzy number. This is because a near-zero fuzzy number cannot be defined in the form of  $(m, \alpha, \beta)$ , unlike a positive or negative fuzzy number could. Therefore, a new form of multiplication arithmetic operator has been introduced by [24] which adapted the system of min-max function.

**Definition 7.** [24] *The product of two fuzzy numbers  $\tilde{M} = (m, \alpha, \beta)$  and  $\tilde{N} = (n, \gamma, \delta)$ , can be defined as*

$$\tilde{M} \otimes \tilde{N} = (mn, f_1, f_2) \quad (14)$$

where

$$f_1 = mn - \text{Min}((m - \alpha)(n - \gamma), (m - \alpha)(n + \delta)),$$

$$f_2 = \text{Max}((m + \beta)(n - \gamma), (m + \beta)(n + \delta)) - mn.$$

The operator as given in (14) is basically has been initiated based on [25] and [26]. In implementing the multiplication, few times multiplication and comparison are needed, to obtain the minimum and maximum values. Besides that, the operator is only compatible for positive fuzzy number  $\tilde{N}$  as stated in the following Theorem 1.

**Theorem 1.** [24] *Consider an arbitrary fuzzy number  $\tilde{M} = (m, \alpha, \beta)$  and a positive fuzzy number  $\tilde{N} = (n, \gamma, \delta)$ ,*

- i. *If  $\tilde{M}$  is positive, then the following inequalities are satisfied:*

$$0 \leq (m - \alpha)(n - \gamma) \leq (m - \alpha)(n + \delta), \quad (15)$$

$$0 \leq (m + \beta)(n - \gamma) \leq (m + \beta)(n + \delta) \quad (16)$$

- ii. *If  $\tilde{M}$  is negative, then the following inequalities are satisfied:*

$$0 \geq (m - \alpha)(n - \gamma) \geq (m - \alpha)(n + \delta), \quad (17)$$

$$0 \geq (m + \beta)(n - \gamma) \geq (m + \beta)(n + \delta) \quad (18)$$

- iii. *If  $\tilde{M}$  is near zero, then the inequalities in (16) and (17) are satisfied.*

### 2.3. Fundamental concepts of fuzzy Kronecker products and fuzzy Vec-operator

Kronecker products and Vec-operator are the important tools in solving matrix equations. The definitions and theorems of the fuzzy Kronecker products and fuzzy Vec-operator, are provided as follows:

**Definition 8.** [17] *Let  $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$  and  $\tilde{B} = (\tilde{b}_{ij})_{p \times q}$  be fuzzy matrices. Fuzzy Kronecker product is represented as  $\tilde{A} \otimes_k \tilde{B}$ , where*

$$\tilde{A} \otimes_k \tilde{B} = \begin{pmatrix} \tilde{a}_{11}\tilde{B} & \tilde{a}_{12}\tilde{B} & \dots & \tilde{a}_{1n}\tilde{B} \\ \tilde{a}_{21}\tilde{B} & \tilde{a}_{22}\tilde{B} & \dots & \tilde{a}_{2n}\tilde{B} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1}\tilde{B} & \tilde{a}_{m2}\tilde{B} & \dots & \tilde{a}_{mn}\tilde{B} \end{pmatrix} = [\tilde{a}_{ij}\tilde{B}]_{(mp) \times (nq)} \quad (19)$$

**Definition 9.** [17] *Vec-operator of a fuzzy matrix is a linear transformation that converts the fuzzy matrix of  $\tilde{C} = (\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_n)$  into a column vector as*

$$Vec(\tilde{C}) = \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_n \end{pmatrix}. \quad (20)$$

**Theorem 2.** [17] *If  $\tilde{A} = (\tilde{a}_{ij})_{m \times m}$  is a fuzzy matrix, and  $\tilde{U} = (\tilde{u}_{ij})_{p \times p}$  is a unitary fuzzy matrix defined as*

$$\tilde{U} = \begin{pmatrix} (1, 0, 0) & (0, 0, 0) & \dots & (0, 0, 0) \\ (0, 0, 0) & (1, 0, 0) & \dots & (0, 0, 0) \\ \vdots & \vdots & \ddots & \vdots \\ (0, 0, 0) & (0, 0, 0) & \dots & (1, 0, 0) \end{pmatrix}, \quad (21)$$

then

i.  $\tilde{A}\tilde{U} = \tilde{U}\tilde{A} = \tilde{A}$

ii.  $\tilde{U}^T = \tilde{U}$ .

**Definition 10.** [17] *Let  $A = (a_{ij})_{m \times m}$ ,  $B = (b_{ij})_{n \times n}$  and  $X = (x_{ij})_{m \times n}$ , then*

i.  $Vec[\tilde{A}\tilde{X}] = [\tilde{U}_n \otimes_k \tilde{A}]Vec(\tilde{X})$

ii.  $Vec[\tilde{X}\tilde{B}] = [\tilde{B}^T \otimes_k \tilde{U}_m]Vec(\tilde{X})$

iii.  $Vec[\tilde{A}\tilde{X}\tilde{B}] = [\tilde{B}^T \otimes_k \tilde{A}]Vec(\tilde{X})$

iv.  $Vec(\tilde{X}) = [\tilde{U}]Vec(\tilde{X})$

### 3. THEORETICAL DEVELOPMENT

This section demonstrates the establishment of the theoretical foundations which involved some theorems, definitions and corollaries. There are four sections presented, consist of the introduction of a new near-zero positive subtraction operator, a modification of arithmetic multiplication operator, some related properties of FFME  $\tilde{A}\tilde{X}\tilde{B} - \tilde{X} = \tilde{C}$  and also the construction of an associated linear systems.

#### 3.1. Near-zero positive subtraction operator

**Theorem 3.** *Let  $\tilde{M} = (m, \alpha, \beta)$  be a near-zero fuzzy number and  $\tilde{N} = (n, \gamma, \delta)$  is a positive fuzzy number. The subtraction of  $\tilde{M}$  and  $\tilde{N}$  is given by*

$$\tilde{M} \ominus_{np} \tilde{N} = (m - n, \alpha + \delta, \beta - \delta). \quad (22)$$

where  $\beta > \delta$ .

*Proof.* Let  $\beta < \delta$ , then  $\beta - \delta < 0$ , which means that the spread value of  $\beta - \delta$  is negative. This is violated since it is always positive, as mentioned in Definition 4. Thus,  $\beta > \delta$ . □

This new operator is known as a Near-zero positive subtraction operator, denoted as  $\ominus_{np}$ .

#### 3.2. Modification of multiplication arithmetic operators

In this study, fuzzy arithmetic multiplication operator as stated in Definition (7) is modified. The modified multiplication operator provides simpler and direct computation as compared to the previous operators.

**Theorem 4.** Let  $\tilde{M} = (m, \alpha, \beta)$  be a positive, negative or near-zero fuzzy number, and  $\tilde{N} = (n, \gamma, \delta)$  be a positive fuzzy number. Then, the min and max in (14) are given by:

$$\begin{aligned} & \text{Min}[(m - \alpha)(n - \gamma), (m - \alpha)(n + \delta)] \\ &= \begin{cases} (m - \alpha)(n - \gamma) & \text{if } \tilde{M} \geq 0 \\ (m - \alpha)(n + \delta) & \text{if otherwise} \end{cases} \end{aligned} \quad (23)$$

$$\begin{aligned} & \text{Max}[(m + \beta)(n - \gamma), (m + \beta)(n + \delta)] \\ &= \begin{cases} (m + \beta)(n - \gamma) & \text{if } \tilde{M} < 0 \\ (m + \beta)(n + \delta) & \text{if otherwise} \end{cases} \end{aligned} \quad (24)$$

*Proof.* Based on Theorem 1 and realize that  $(n - \gamma) < (n + \delta)$ , then obviously:

- If  $\tilde{M}$  is positive which is  $(m - \alpha) \geq 0$ , both multiplications of  $(m - \alpha)(n - \gamma)$  and  $(m + \beta)(n - \gamma)$  are minimum compared to the multiplications of  $(m - \alpha)(n + \delta)$  and  $(m + \beta)(n + \delta)$  respectively. However, if  $(m - \alpha) < (m + \beta)$ , then  $(m - \alpha)(n - \gamma)$  is minimum.  
On the other hand, since both multiplications of  $(m - \alpha)(n + \delta)$  and  $(m + \beta)(n + \delta)$  are maximum compared to the multiplication of  $(m - \alpha)(n - \gamma)$  and  $(m + \beta)(n - \gamma)$  respectively, but since  $(m + \beta) > (m - \alpha)$ , thus the maximum value is  $(m + \beta)(n + \delta)$ .
- If  $\tilde{M}$  is negative which is  $(m - \alpha) < 0$ , both multiplications of  $(m - \alpha)(n + \delta)$  and  $(m + \beta)(n + \delta)$  are minimum compared to the multiplication of  $(m - \alpha)(n - \gamma)$  and  $(m + \beta)(n - \gamma)$  respectively. From that, since  $(m - \alpha) < (m + \beta)$ , then  $(m - \alpha)(n + \delta)$  is minimum.  
On the other hand, since both  $(m - \alpha)(n - \gamma)$  and  $(m + \beta)(n - \gamma)$  are maximum compared to the multiplication of  $(m - \alpha)(n + \delta)$  and  $(m + \beta)(n + \delta)$  respectively, but  $(m + \beta) > (m - \alpha)$  thus the maximum value is  $(m + \beta)(n - \gamma)$ .
- If  $\tilde{M}$  is near-zero which is  $(m - \alpha) \leq 0 \leq (\beta + m)$ , based on the inequities in (16) and (17), then obviously  $(m - \alpha)(n + \delta)$  is minimum, whereas  $(m + \beta)(n + \delta)$  is maximum.

□

From Theorem 4 and (14), the modified multiplication arithmetic operators are defined in the following theorem.

**Theorem 5.** Let  $\tilde{M} = (m, \alpha, \beta)$  be a positive, negative or near-zero fuzzy number, and  $\tilde{N} = (n, \gamma, \delta)$  be a positive fuzzy number, then the multiplication of  $\tilde{M} \otimes \tilde{N}$  is defined as follows:

1. If  $\tilde{M}$  is positive, then

$$\tilde{M} \otimes \tilde{N} = (m, \alpha, \beta) \otimes (n, \gamma, \delta) \cong (mn, n\alpha + (m - \alpha)\gamma, n\beta + (m + \beta)\delta) \quad (25)$$

2. If  $\tilde{M}$  is negative, then

$$\tilde{M} \otimes \tilde{N} = (m, \alpha, \beta) \otimes (n, \gamma, \delta) \cong (mn, n\alpha - (m - \alpha)\delta, n\beta - (m + \beta)\gamma) \quad (26)$$

3. If  $\tilde{M}$  is near-zero, then

$$\tilde{M} \otimes \tilde{N} = (m, \alpha, \beta) \otimes (n, \gamma, \delta) \cong (mn, n\alpha - (m - \alpha)\delta, n\beta + (m + \beta)\delta) \quad (27)$$

*Proof.* By considering the Corollary 4, and applying it to (14), thus:

1. For  $\tilde{M}$  is positive,

$$\begin{aligned} \tilde{M} \otimes \tilde{N} &= (mn, mn - (m - \alpha)(n - \gamma), (m + \beta)(n + \delta) - mn) \\ &= (mn, mn - mn + m\gamma + \alpha n - \alpha\delta, mn + m\delta + \beta n + \beta\delta - mn) \\ &= (mn, m\gamma + \alpha n - \alpha\delta, m\delta + \beta n + \beta\delta) \\ &= (mn, \alpha n + (m - \alpha)\gamma, \beta n + (m + \beta)\delta) \end{aligned} \quad (28)$$

2. For  $\tilde{M}$  is negative,

$$\begin{aligned}\tilde{M} \otimes \tilde{N} &= (mn, mn - (m - \alpha)(n + \delta), (m + \beta)(n - \gamma) - mn) \\ &= (mn, mn - mn - m\delta + \alpha n + \alpha\delta, mn - m\gamma + \beta n - \beta\gamma - mn) \\ &= (mn, -m\delta + \alpha n + \alpha\delta, -m\gamma + \beta n - \beta\gamma) \\ &= (mn, \alpha n - (m - \alpha)\delta, \beta n - (m + \beta)\gamma)\end{aligned}\quad (29)$$

3. For  $\tilde{M}$  is near-zero,

$$\begin{aligned}\tilde{M} \otimes \tilde{N} &= (mn, mn - (m - \alpha)(n + \delta), (m + \beta)(n + \delta) - mn) \\ &= (mn, mn - mn - m\delta + \alpha n + \alpha\delta, mn + m\delta + \beta n + \beta\delta - mn) \\ &= (mn, -m\delta + \alpha n + \alpha\delta, m\delta + \beta n + \beta\delta) \\ &= (mn, \alpha n - (m - \alpha)\delta, \beta n + (m + \beta)\delta)\end{aligned}\quad (30)$$

Since (25) to (27) are shown, hence the theorem is proved. □

**Corollary 1.** Let  $\tilde{M} = (m, \alpha, \beta)$  be a positive, negative or near-zero fuzzy number, and  $\tilde{N} = (n, \gamma, \delta)$  be a positive fuzzy number:

1. If  $\tilde{M}$  is positive, then the multiplication of  $\tilde{M} \otimes \tilde{N}$  is positive, such that  $mn - (n\alpha + (m - \alpha)\gamma) > 0$ .
2. If  $\tilde{M}$  is negative, then the multiplication of  $\tilde{M} \otimes \tilde{N}$  is negative, such that  $(n\beta - (m + \beta)\gamma) + mn < 0$ .
3. If  $\tilde{M}$  is near-zero, then the multiplication of  $\tilde{M} \otimes \tilde{N}$  is near-zero, such that  $mn - (n\alpha - (m - \alpha)\delta) < 0 < (n\beta + (m + \beta)\delta) + mn$ .

*Proof.* The multiplication of  $\tilde{M} \otimes \tilde{N}$  in Theorem 5 must satisfy the Definition 5, where

1. If  $\tilde{M}$  is positive, then  $mn - (n\alpha + (m - \alpha)\gamma) > 0$  which is

$$\begin{aligned}mn - (n\alpha + (m - \alpha)\gamma) &= mn - n\alpha - m\gamma + \alpha\gamma \\ &= (m - \alpha)n - (m - \alpha)\gamma \\ &= (m - \alpha)(n - \gamma)\end{aligned}\quad (31)$$

Since both  $(m - \alpha)$  and  $(n - \gamma)$  are  $> 0$ , then  $mn - (n\alpha + (m - \alpha)\gamma) > 0$ .

2. If  $\tilde{M}$  is negative, then  $(n\beta - (m + \beta)\gamma) + mn < 0$  which is

$$\begin{aligned}(n\beta - (m + \beta)\gamma) + mn &= n\beta - m\gamma - \beta\gamma + mn \\ &= (m + \beta)n - (m + \beta)\gamma \\ &= (m + \beta)(n - \gamma)\end{aligned}\quad (32)$$

Since  $(m + \beta) < 0$  and  $(n - \gamma) > 0$ , then  $mn + (n\beta - (m + \beta)\gamma) < 0$ .

3. If  $\tilde{M}$  is near-zero, then  $mn - (n\alpha - (m - \alpha)\delta) < 0 < (n\beta + (m + \beta)\delta) + mn$  which is

$$\begin{aligned}mn - (n\alpha - (m - \alpha)\delta) &= mn - n\alpha + m\delta - \alpha\delta \\ &= (m - \alpha)n + (m - \alpha)\delta \\ &= (m - \alpha)(n + \delta)\end{aligned}\quad (33)$$

Since  $(m - \alpha) < 0$  and  $(n + \delta) > 0$ , then  $mn - (n\alpha - (m - \alpha)\delta) < 0$ .

On the other hand,

$$\begin{aligned}(n\beta + (m + \beta)\delta) + mn &= n\beta + m\delta + \beta\delta + mn \\ &= (m + \beta)n + (m + \beta)\delta \\ &= (m + \beta)(n + \delta)\end{aligned}\quad (34)$$

Since  $(m + \beta) > 0$  and  $(n + \delta) > 0$ , then  $(n\beta + (m + \beta)\delta) + mn > 0$ .

After all the conditions are satisfied, then the corollary is proved. □

### 3.3. Related properties of FFME $\tilde{A}\tilde{X}\tilde{B} - \tilde{X} = \tilde{C}$

The definition of FFME  $\tilde{A}\tilde{X}\tilde{B} - \tilde{X} = \tilde{C}$  is given as follows:

**Definition 11.** The matrix equation

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1m} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \cdots & \tilde{a}_{mm} \end{pmatrix} \otimes \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{x}_{m1} & \tilde{x}_{m2} & \cdots & \tilde{x}_{mn} \end{pmatrix} \otimes \begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \cdots & \tilde{b}_{1n} \\ \tilde{b}_{21} & \tilde{b}_{22} & \cdots & \tilde{b}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{b}_{n1} & \tilde{b}_{n2} & \cdots & \tilde{b}_{nm} \end{pmatrix} \ominus \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{x}_{m1} & \tilde{x}_{m2} & \cdots & \tilde{x}_{mn} \end{pmatrix} = \begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \cdots & \tilde{c}_{1n} \\ \tilde{c}_{21} & \tilde{c}_{22} & \cdots & \tilde{c}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{c}_{m1} & \tilde{c}_{m2} & \cdots & \tilde{c}_{mn} \end{pmatrix} \quad (35)$$

can also be represented as

$$\tilde{A}\tilde{X}\tilde{B} - \tilde{X} = \tilde{C} \quad (36)$$

where  $\tilde{A} = (a_{ij})$ ,  $1 \leq i, j \leq n$ ,  $\tilde{B} = (b_{ij})$ ,  $1 \leq i, j \leq m$ , the right hand side matrix  $\tilde{C} = (c_{ij})$ ,  $1 \leq i \leq n, 1 \leq j \leq m$  is the fuzzy matrices, and  $\tilde{X} = (x_{ij})$ ,  $1 \leq i \leq n, 1 \leq j \leq m$  is an unknown fuzzy matrix.

There is a special criterion related to the order of matrix coefficients for FFME  $\tilde{A}\tilde{X}\tilde{B} - \tilde{X} = \tilde{C}$ .

**Remark 2.** Let  $\tilde{A}\tilde{X}\tilde{B} - \tilde{X} = \tilde{C}$  be an FFME, where the fuzzy coefficient of  $\tilde{A}$  and  $\tilde{B}$  must be any square matrices.

**Example 1.** If  $\tilde{A}$  and  $\tilde{B}$  are non-square matrices with any appropriate orders of  $\tilde{A}_{r \times p}$  and  $\tilde{B}_{q \times s}$ , and the solution is  $\tilde{X}_{p \times q}$ , then

$$\tilde{A}_{r \times p} \tilde{X}_{p \times q} \tilde{B}_{q \times s} - \tilde{X}_{p \times q} = \tilde{A}\tilde{X}\tilde{B}_{r \times s} - \tilde{X}_{p \times q}.$$

However, the subtraction of  $\tilde{A}\tilde{X}\tilde{B}_{r \times s}$  and  $\tilde{X}_{p \times q}$  is not possible due to the different order. Thus, in all cases,  $\tilde{A}$  and  $\tilde{B}$  in FFME  $\tilde{A}\tilde{X}\tilde{B} - \tilde{X} = \tilde{C}$  must be square matrices.

### 3.4. Construction of an associated linear system

**Definition 12.** Consider a fully fuzzy linear system (FFLS) in the form of

$$\tilde{S}\tilde{X} = \tilde{C} \quad (37)$$

where  $\tilde{S} = (m, \alpha, \beta)$ ,  $\tilde{X} = (n, \gamma, \delta)$  and  $\tilde{C} = (C, G, H)$ , which is equivalent to

$$\sum_{j=1, \dots, n}^n (m_{ij}, \alpha_{ij}, \beta_{ij}) \otimes (n_j, \gamma_j, \delta_j) = (C_i, G_i, H_i). \quad (38)$$

According to the new multiplication arithmetic operators stated in Theorem 5, the FFLS can be transformed in a form of a crisp linear system, called as the ALS.

**Definition 13.** Let  $\tilde{S} = (m, \alpha, \beta)$  be a positive, negative or near-zero fuzzy number,  $\tilde{X} = (n, \gamma, \delta)$  be a positive fuzzy number and  $\tilde{C} = (C, G, H)$  be any form of fuzzy numbers, based on the multiplication arithmetic operators in Theorem 5. Then, three forms of ALS are obtained, such that:

- If  $\tilde{S}$  is positive,

$$\begin{cases} mn & = C \\ \alpha n + (m - \alpha)\gamma & = G \\ \beta n + (m + \beta)\delta & = H \end{cases}$$

$$\left( \begin{array}{c|c|c} m & 0 & 0 \\ \alpha & (m - \alpha) & 0 \\ \beta & 0 & (m + \beta) \end{array} \right) \begin{pmatrix} n \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} C \\ G \\ H \end{pmatrix} \quad (39)$$



- If  $\tilde{S}$  is negative,

$$\begin{cases} mn & = C \\ \alpha n - (m - \alpha)\delta & = G \\ \beta n - (m + \beta)\gamma & = H \end{cases}$$

which can be represented as

$$\left( \begin{array}{c|cc} m & 0 & 0 \\ \alpha & 0 & -(m - \alpha) \\ \beta & -(m + \beta) & 0 \end{array} \right) \begin{pmatrix} n \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} C \\ G \\ H \end{pmatrix} \quad (40)$$

- If  $\tilde{S}$  is near-zero,

$$\begin{cases} mn & = C \\ \alpha n - (m - \alpha)\delta & = G \\ \beta n + (m + \beta)\delta & = H \end{cases}$$

which can be represented as

$$\left( \begin{array}{c|cc} m & 0 & 0 \\ \alpha & 0 & -(m - \alpha) \\ \beta & 0 & (m + \beta) \end{array} \right) \begin{pmatrix} n \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} C \\ G \\ H \end{pmatrix} \quad (41)$$

By applying the concept of union sets as stated in Definition 2, these three ALS block matrices in (39), (40) and (41) can be combined into a single ALS as illustrated in Definition 14.

**Definition 14.** Let  $\tilde{S}\tilde{X} = \tilde{C}$  be a FFLS, where the fuzzy coefficients  $\tilde{S}$  and  $\tilde{C}$  are arbitrary fuzzy numbers and  $\tilde{X}$  be a positive fuzzy solution. ALS is represented as

$$\begin{cases} mn & = C \\ \alpha n + (m - \alpha)\gamma - (m - \alpha)\delta & = G \\ \beta n - (m + \beta)\gamma + (m + \beta)\delta & = H \end{cases} \quad (42)$$

which can be written in the matrix form of

$$\left( \begin{array}{c|cc} m & 0 & 0 \\ \alpha & (m - \alpha) & -(m - \alpha) \\ \beta & -(m + \beta) & (m + \beta) \end{array} \right) \begin{pmatrix} n \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} C \\ G \\ H \end{pmatrix} \quad (43)$$

where

$$m = (m_{ij})_{m \times n} = \begin{pmatrix} m_{11} & \dots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{m1} & \dots & m_{mn} \end{pmatrix}, \alpha = (\alpha_{ij})_{m \times n} = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \dots & \alpha_{mn} \end{pmatrix},$$

$$\beta = (\beta_{ij})_{m \times n} = \begin{pmatrix} \beta_{11} & \dots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{m1} & \dots & \beta_{mn} \end{pmatrix},$$

$$n = \begin{pmatrix} n_1 \\ \vdots \\ n_n \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}, \quad \delta = \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_n \end{pmatrix},$$

and

$$C = \begin{pmatrix} C_1 \\ \vdots \\ C_m \end{pmatrix}, \quad G = \begin{pmatrix} G_1 \\ \vdots \\ G_m \end{pmatrix}, \quad H = \begin{pmatrix} H_1 \\ \vdots \\ H_m \end{pmatrix}.$$

This ALS can be denoted as

$$SX = C. \quad (44)$$

However, the matrix  $S$  in (43) is always inconsistent since  $|S| = 0$ , which is proved in the following theorem:

**Theorem 6.** Let  $S$  be a coefficient of an ALS. The matrix  $S$  is singular or  $|S| = 0$ , when  $|m| = 0$  or

$$\left| \begin{array}{c|c} (m - \alpha) & -(m - \alpha) \\ \hline -(m + \beta) & (m + \beta) \end{array} \right| = 0.$$

*Proof.* Let

$$S = \left( \begin{array}{c|c|c} m & 0 & 0 \\ \hline \alpha & (m - \alpha) & -(m - \alpha) \\ \hline \beta & -(m + \beta) & (m + \beta) \end{array} \right)$$

The singularity of  $S$  can be determined from the following procedure, which is based on Remark 1.

$$\begin{aligned} |S| &= \det \left[ \begin{array}{c|c|c} (m - \alpha) - \alpha(m)^{-1}(0) & -(m - \alpha) - \alpha(m)^{-1}(0) \\ \hline -(m + \beta) - \beta(m)^{-1}(0) & (m + \beta) - \beta(m)^{-1}(0) \end{array} \right] \times \det[m] \\ &= \det \left[ \begin{array}{c|c} (m - \alpha) & -(m - \alpha) \\ \hline -(m + \beta) & (m + \beta) \end{array} \right] \times \det[m] \end{aligned}$$

From this, if  $|m| = 0$ , then obviously matrix  $S$  is singular. On the other hand, if  $|m| \neq 0$ , but

$$\left| \begin{array}{c|c} (m - \alpha) & -(m - \alpha) \\ \hline -(m + \beta) & (m + \beta) \end{array} \right| = 0, \text{ hence, matrix } S \text{ is singular.} \quad \square$$

**Remark 3.** There are two possibilities that make the determinant of  $\left[ \begin{array}{c|c} (m - \alpha) & -(m - \alpha) \\ \hline -(m + \beta) & (m + \beta) \end{array} \right] = 0$ , which are:

i. At least one block matrix in both diagonal and anti-diagonal have all zeroes in a row, such that:

$$\left( \begin{array}{cc|cc} 0 & 0 & a & b \\ 0 & 0 & c & d \\ \hline 0 & 0 & e & f \\ 0 & 0 & g & h \end{array} \right)$$

ii. The  $i^{\text{th}}$  row or  $j^{\text{th}}$  column of a matrix is a multiple of another row or column, such that:

$$\left( \begin{array}{cc|cc} a & -b & -a & b \\ c & d & -c & -d \\ \hline -e & -f & e & f \\ -g & -h & g & h \end{array} \right)$$

In order to avoid the inconsistency of the solution, the ALS in (43) has been improvised to be in the following form as stated in the next theorem.

**Definition 15.** Let  $\tilde{S}\tilde{X} = \tilde{C}$  be a FFLS such that  $\tilde{S} = (m, \alpha, \beta)$ ,  $\tilde{X} = (n, \gamma, \delta)$  and  $\tilde{C} = (C, G, H)$ , with solution  $\tilde{X}$  as a positive fuzzy number. Then the ALS of  $SX = C$  is written as:

$$\left( \begin{array}{c|c|c} m & 0 & 0 \\ \hline \alpha & (m - \alpha)^+ & -(m - \alpha)^- \\ \hline \beta & -(m + \beta)^- & (m + \beta)^+ \end{array} \right) \begin{pmatrix} n \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} C \\ G \\ H \end{pmatrix} \quad (45)$$

where  $(m - \alpha)^+$  and  $(m + \beta)^+$  contain the positive elements of  $(m - \alpha)$  and  $(m + \beta)$  respectively, while the negative elements are replaced by zero values. Similarly,  $(m - \alpha)^-$  and  $(m + \beta)^-$  contain the negative elements of  $(m - \alpha)$  and  $(m + \beta)$  respectively, while the positive elements are replaced by zero values.

The solution obtained could be justified as a strong or weak fuzzy solution. According to [12, 27, 28], the definition of strong and weak fuzzy solutions is written as follows:

**Definition 16.** A positive fuzzy solution  $\tilde{X} = (n, \gamma, \delta)$  of FFME in (1) is called a strong fuzzy solution if  $\gamma > 0$  and  $\delta > 0$ , otherwise it is called a weak fuzzy solution.

#### 4. ALGORITHM FOR SOLVING $\tilde{A}\tilde{X}\tilde{B} - \tilde{X} = \tilde{C}$

There are three steps involved in constructing the algorithm. It begins with a conversion of FFME to its equivalent FFLS, then followed by transforming the FFLS to the crisp form of matrices. From that, the ALS is formed. Finally, the solution is obtained using the direct inverse method.

##### Step 1. Converting the FFME into FFLS.

Firstly, the FFME is converted to a simpler form of equation known as FFLS. The conversion is based on the fuzzy Kronecker product and *Vec*-operator. Taking *Vec*-operator for both sides of (1), we have

$$Vec(\tilde{A}\tilde{X}\tilde{B} - \tilde{X}) = Vec(\tilde{C}),$$

then,

$$Vec(\tilde{A}\tilde{X}\tilde{B}) - Vec(\tilde{X}) = Vec(\tilde{C}). \quad (46)$$

As we assumed that one of the coefficients,  $\tilde{A}$  or  $\tilde{B}$  in the FFME of (1) is a near-zero fuzzy matrix, then based on Corollary 1,  $Vec(\tilde{A}\tilde{X}\tilde{B})$  is also a near-zero fuzzy matrix. Next, based on Theorem 10

$$[(\tilde{B}^T \otimes_k \tilde{A})] Vec(\tilde{X}) - [\tilde{U}] Vec(\tilde{X}) = Vec(\tilde{C}) \quad (47)$$

then,

$$[(\tilde{B}^T \otimes_k \tilde{A}) - \tilde{U}] Vec(\tilde{X}) = Vec(\tilde{C}) \quad (48)$$

Suppose  $[(\tilde{B}^T \otimes_k \tilde{A}) - \tilde{U}] = \tilde{S}$ , where  $\tilde{S} = (F, M, N)$ ,  $Vec(\tilde{X}) = (m^x, \alpha^x, \beta^x)$  and  $Vec(\tilde{C}) = (m^c, \alpha^c, \beta^c)$ , then (48) can be written as

$$\tilde{S}Vec(\tilde{X}) = Vec(\tilde{C}).$$

##### Step 2. Converting the FFLS to the crisp form of matrices.

In this step, both coefficients of FFLS,  $\tilde{S}$  and  $\tilde{C}$ , are converted to its corresponding crisp matrices.

Considering that  $\tilde{S}Vec(\tilde{X}) = Vec(\tilde{C})$  in the following form,

$$\sum_{j=1, \dots, n} (m_{ij}^s, \alpha_{ij}^s, \beta_{ij}^s) \otimes (m_j^x, \alpha_j^x, \beta_j^x) = (m_i^c, \alpha_i^c, \beta_i^c). \quad (49)$$

Then,

$$m^s = (m_{ij}^s)_{m \times n} = \begin{pmatrix} m_{11}^s & \dots & m_{1n}^s \\ \vdots & \ddots & \vdots \\ m_{m1}^s & \dots & m_{mn}^s \end{pmatrix}, \alpha^s = (\alpha_{ij}^s)_{m \times n} = \begin{pmatrix} \alpha_{11}^s & \dots & \alpha_{1n}^s \\ \vdots & \ddots & \vdots \\ \alpha_{m1}^s & \dots & \alpha_{mn}^s \end{pmatrix},$$

$$\beta^s = (\beta_{ij}^s)_{n \times n} = \begin{pmatrix} \beta_{11}^s & \dots & \beta_{1n}^s \\ \vdots & \ddots & \vdots \\ \beta_{n1}^s & \dots & \beta_{nn}^s \end{pmatrix},$$

$$m^x = \begin{pmatrix} m_1^x \\ \vdots \\ m_n^x \end{pmatrix}, \alpha^x = \begin{pmatrix} \alpha_1^x \\ \vdots \\ \alpha_n^x \end{pmatrix}, \beta^x = \begin{pmatrix} \beta_1^x \\ \vdots \\ \beta_n^x \end{pmatrix},$$

and

$$m^c = \begin{pmatrix} m_1^c \\ \vdots \\ m_m^c \end{pmatrix}, \quad \alpha^c = \begin{pmatrix} \alpha_1^c \\ \vdots \\ \alpha_m^c \end{pmatrix}, \quad \beta^c = \begin{pmatrix} \beta_1^c \\ \vdots \\ \beta_m^c \end{pmatrix}$$

### Step 3. Forming the ALS.

Based on the crisp matrices obtained in Step 2, the values for  $(m^s - \alpha^s)$ ,  $(m^s + \beta^s)$ ,  $(m^s - \alpha^s)^+$ ,  $(m^s - \alpha^s)^-$ ,  $(m^s + \beta^s)^+$  and  $(m^s + \beta^s)^-$  are determined. Then, the ALS of (45) is formed.

### Step 4. Obtaining the final solution.

In obtaining the final solution, a direct inverse method is applied to the ALS.

## 5. NUMERICAL EXAMPLE

**Example 2.** Consider the following FFME of  $\tilde{A}\tilde{X}\tilde{B} - \tilde{X} = \tilde{C}$ .

$$\begin{pmatrix} (18, 2, 16) & (10, 2, 7) & (15, 1, 5) \\ (19, 8, 10) & (9, 5, 7) & (8, 2, 12) \\ (7, 2, 17) & (10, 5, 9) & (6, 3, 5) \end{pmatrix} \otimes \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \\ \tilde{x}_{31} & \tilde{x}_{32} \end{pmatrix} \otimes \begin{pmatrix} (-10, 3, 12) & (7, 4, 11) \\ (8, 5, 16) & (-10, 6, 12) \end{pmatrix} - \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \\ \tilde{x}_{31} & \tilde{x}_{32} \end{pmatrix} \\ = \begin{pmatrix} (-333, 11887, 28754) & (-1710, 15217, 21983) \\ (5, 11484, 25602) & (-1750, 13722, 20295) \\ (-180, 10012, 22202) & (-856, 12536, 17038) \end{pmatrix}$$

where coefficient  $\tilde{A}$  is positive, while  $\tilde{B}$  is a near-zero fuzzy number.

**Solution:**

**Step 1.** Converting the FFME into FFLS by fuzzy Kronecker product and *Vec*-operator.

$$\begin{pmatrix} \tilde{B}^T \otimes_k \tilde{A} \end{pmatrix} - \tilde{U} \\ = \begin{pmatrix} (-181, 262, 248) & (-100, 121, 134) & (-150, 110, 190) & (144, 96, 672) & (80, 56, 328) & (120, 78, 360) \\ (-190, 187, 248) & (-91, 118, 122) & (-80, 180, 120) & (152, 119, 544) & (72, 60, 312) & (64, 46, 416) \\ (-70, 242, 118) & (-100, 147, 138) & (-61, 83, 82) & (56, 41, 520) & (80, 65, 376) & (48, 39, 216) \\ (126, 78, 486) & (70, 46, 236) & (105, 63, 255) & (-181, 364, 248) & (-100, 172, 134) & (-150, 170, 190) \\ (133, 100, 389) & (63, 51, 225) & (56, 38, 304) & (-190, 274, 248) & (-91, 166, 122) & (-80, 240, 120) \\ (49, 34, 383) & (70, 55, 272) & (42, 33, 156) & (-70, 314, 118) & (-100, 204, 138) & (-61, 116, 82) \end{pmatrix}.$$

From that, the FFME can be written in the form of FFLS  $[(\tilde{B}^T \otimes_k \tilde{A}) - \tilde{U}]Vec(\tilde{X}) = Vec(\tilde{C})$ , which is given as follows:

$$\begin{pmatrix} (-181, 262, 248) & (-100, 121, 134) & (-150, 110, 190) & (144, 96, 672) & (80, 56, 328) & (120, 78, 360) \\ (-190, 187, 248) & (-91, 118, 122) & (-80, 180, 120) & (152, 119, 544) & (72, 60, 312) & (64, 46, 416) \\ (-70, 242, 118) & (-100, 147, 138) & (-61, 83, 82) & (56, 41, 520) & (80, 65, 376) & (48, 39, 216) \\ (126, 78, 486) & (70, 46, 236) & (105, 63, 255) & (-181, 364, 248) & (-100, 172, 134) & (-150, 170, 190) \\ (133, 100, 389) & (63, 51, 225) & (56, 38, 304) & (-190, 274, 248) & (-91, 166, 122) & (-80, 240, 120) \\ (49, 34, 383) & (70, 55, 272) & (42, 33, 156) & (-70, 314, 118) & (-100, 204, 138) & (-61, 116, 82) \end{pmatrix} \\ \begin{pmatrix} (m_{1,1}^x, \alpha_{1,1}^x, \beta_{1,1}^x) \\ (m_{2,1}^x, \alpha_{2,1}^x, \beta_{2,1}^x) \\ (m_{3,1}^x, \alpha_{3,1}^x, \beta_{3,1}^x) \\ (m_{1,2}^x, \alpha_{1,2}^x, \beta_{1,2}^x) \\ (m_{2,2}^x, \alpha_{2,2}^x, \beta_{2,2}^x) \\ (m_{3,2}^x, \alpha_{3,2}^x, \beta_{3,2}^x) \end{pmatrix} = \begin{pmatrix} (-333, 11887, 28754) \\ (5, 11484, 25602) \\ (-180, 10012, 22202) \\ (-1710, 15217, 21983) \\ (-1750, 13722, 20295) \\ (-856, 12536, 17038) \end{pmatrix}.$$

**Step 2.** All the coefficients are written in the crisp form as follows:

$$m^s = \begin{pmatrix} -181 & -100 & -150 & 144 & 80 & 120 \\ -190 & -91 & -80 & 152 & 72 & 64 \\ -70 & -100 & -61 & 56 & 80 & 48 \\ 126 & 70 & 105 & -181 & -100 & -150 \\ 133 & 63 & 56 & -190 & -91 & -80 \\ 49 & 70 & 42 & -70 & -100 & -61 \end{pmatrix},$$



**Step 4:** Finally, the solution is obtained by the direct inverse method as follows:

$$X = \begin{pmatrix} m_{1,1}^x \\ m_{2,1}^x \\ m_{3,1}^x \\ m_{1,2}^x \\ m_{2,2}^x \\ m_{3,2}^x \\ \alpha_{1,1}^x \\ \alpha_{2,1}^x \\ \alpha_{3,1}^x \\ \alpha_{1,2}^x \\ \alpha_{2,2}^x \\ \alpha_{3,2}^x \\ \beta_{1,1}^x \\ \beta_{2,1}^x \\ \beta_{3,1}^x \\ \beta_{1,2}^x \\ \beta_{2,2}^x \\ \beta_{3,2}^x \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \\ 10 \\ 14 \\ 8 \\ 7 \\ 2 \\ 2 \\ 4 \\ 3 \\ 3 \\ 2 \\ 4 \\ 12 \\ 2 \\ 5 \\ 7 \\ 3 \end{pmatrix} \quad \text{or} \quad X = \begin{pmatrix} (m_{1,1}^x) \\ (m_{2,1}^x) \\ (m_{3,1}^x) \\ (m_{1,2}^x) \\ (m_{2,2}^x) \\ (m_{3,2}^x) \\ (\alpha_{1,1}^x) \\ (\alpha_{2,1}^x) \\ (\alpha_{3,1}^x) \\ (\alpha_{1,2}^x) \\ (\alpha_{2,2}^x) \\ (\alpha_{3,2}^x) \\ (\beta_{1,1}^x) \\ (\beta_{2,1}^x) \\ (\beta_{3,1}^x) \\ (\beta_{1,2}^x) \\ (\beta_{2,2}^x) \\ (\beta_{3,2}^x) \end{pmatrix} = \begin{pmatrix} (9) \\ (7) \\ (10) \\ (14) \\ (8) \\ (7) \\ (2) \\ (2) \\ (4) \\ (3) \\ (3) \\ (2) \\ (4) \\ (12) \\ (2) \\ (5) \\ (7) \\ (3) \end{pmatrix}.$$

Hence, the solution obtained is a strong fuzzy solution of

$$\tilde{X} = \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \\ \tilde{x}_{31} & \tilde{x}_{32} \end{pmatrix} = \begin{pmatrix} (9, 2, 4) & (14, 3, 5) \\ (7, 2, 12) & (8, 3, 7) \\ (10, 4, 2) & (7, 2, 3) \end{pmatrix}. \quad (50)$$

#### Verification of the solution

The solution is verified by substituting the solution  $\tilde{X}$  obtained in (50) to the left hand side of Example 2.

$$\begin{aligned} \tilde{A}\tilde{X}\tilde{B} &= \begin{pmatrix} (18, 2, 16) & (10, 2, 7) & (15, 1, 5) \\ (19, 8, 10) & (9, 5, 7) & (8, 2, 12) \\ (7, 2, 17) & (10, 5, 9) & (6, 3, 5) \end{pmatrix} \otimes \begin{pmatrix} (9, 2, 4) & (14, 3, 5) \\ (7, 2, 12) & (8, 3, 7) \\ (10, 4, 2) & (7, 2, 3) \end{pmatrix} \otimes \begin{pmatrix} (-10, 3, 12) & (7, 4, 11) \\ (8, 5, 16) & (-10, 6, 12) \end{pmatrix} \\ &= \begin{pmatrix} (-324, 11883, 28758) & (-1696, 15212, 21988) \\ (12, 11472, 25614) & (-1742, 13715, 20302) \\ (-170, 10010, 22204) & (-849, 12533, 17041) \end{pmatrix} \end{aligned}$$

After that, since the fuzzy matrix  $\tilde{A}\tilde{X}\tilde{B}$  is near-zero, while  $\tilde{X}$  is positive, the subtraction operator defined in Definition 3 is applied to implement the subtraction of  $\tilde{A}\tilde{X}\tilde{B} - \tilde{X}$ .

$$\begin{aligned} \tilde{A}\tilde{X}\tilde{B} - \tilde{X} &= \begin{pmatrix} (-324, 11883, 28758) & (-1696, 15212, 21988) \\ (12, 11472, 25614) & (-1742, 13715, 20302) \\ (-170, 10010, 22204) & (-849, 12533, 17041) \end{pmatrix} - \begin{pmatrix} (9, 2, 4) & (14, 3, 5) \\ (7, 2, 12) & (8, 3, 7) \\ (10, 4, 2) & (7, 2, 3) \end{pmatrix} \\ &= \begin{pmatrix} (-333, 11887, 28754) & (-1710, 15217, 21983) \\ (5, 11484, 25602) & (-1750, 13722, 20295) \\ (-180, 10012, 22202) & (-856, 12536, 17038) \end{pmatrix} \end{aligned}$$

Hence, the solution is verified. However, if the subtraction of  $\tilde{A}\tilde{X}\tilde{B} - \tilde{X}$  is implemented using the existing subtraction operator in (10), the following solution is obtained, which does not satisfy the right-hand side of the matrix.

$$\begin{aligned} \tilde{A}\tilde{X}\tilde{B} - \tilde{X} &= \begin{pmatrix} (-324, 11883, 28758) & (-1696, 15212, 21988) \\ (12, 11472, 25614) & (-1742, 13715, 20302) \\ (-170, 10010, 22204) & (-849, 12533, 17041) \end{pmatrix} - \begin{pmatrix} (9, 2, 4) & (14, 3, 5) \\ (7, 2, 12) & (8, 3, 7) \\ (10, 4, 2) & (7, 2, 3) \end{pmatrix} \\ &= \begin{pmatrix} (-333, 11887, 28760) & (-1710, 15217, 21991) \\ (5, 11484, 25616) & (-1750, 13722, 20305) \\ (-180, 10012, 22208) & (-856, 12536, 17043) \end{pmatrix} \end{aligned}$$

## 6. CONCLUSION

This paper provides a modification of new subtraction and multiplication operators in order to solve the FFME of  $\tilde{A}\tilde{X}\tilde{B}-\tilde{X}=\tilde{C}$ , where either coefficient  $\tilde{A}$  or  $\tilde{B}$  is a near-zero fuzzy matrix. A modification is needed since the existing operators are not well-equipped to deal with the near-zero fuzzy numbers. The proposed algorithm involves the transformation of FFME to FFLS by utilizing Kronecker product and *Vec*-operator. Subsequently, a new associated linear system is established based on the new modification of multiplication operators. Finally, the solution is easily obtained using the direct inverse method. As a result, the proposed algorithm has provides a significant approach with fewer restrictions in terms of fuzzy numbers, regardless of the size of the matrix equations.

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