

A numerical-analytical iterative method for solving an electrical oscillator equation

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ABSTRACT

Self-excited oscillation problem occurring from a triode electrical circuit has been modelled by van der Pol. Until now, the exact solution to the van der Pol equation is not available. This paper focuses on finding a new method for solving the van der Pol equation simply and accurately. There exists several approximate iterative methods available in the literature for solving the van der Pol equation, such as, the successive approximation method. The successive approximation method is simple, but inaccurate for large time values. In this paper, we propose a new variant of numerical-analytical method, which is simple but accurate, for solving the van der Pol equation. Our new variant of numerical-analytical method solves the van der Pol equation from its equivalent system of first order ordinary differential equations. Our strategy leads to a simple implementation of the numerical-analytical method in the multistage way. Furthermore, computational experiments show that our proposed method is accurate for large sizes of time domain in solving the van der Pol equation.

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1. INTRODUCTION

Mathematical models have been used for simulating various problems in the areas of telecommunication, computing, electronics and control [1]-[5]. The solution to the mathematical model represents the solution to the real problem. As a large number of mathematical models do not have analytical exact solutions in general, numerical methods are usually used to obtain approximate solutions [6]-[9].

Other than numerical methods, analytical approaches such as the successive approximation method provide approximate solutions [10]-[16]. The successive approximation method can be considered as a special case of variational iteration methods, which have been used to solve problems in biology [17], [18], engineering [19]-[22], and others [23]-[28]. In this work, we shall combine numerical and analytical methods to form a new numerical-analytical method.

In this paper, we shall solve the van der Pol equation arising from a triode electrical circuit problem. The van der Pol equation models self-excited oscillations of the circuit [29]. We limit this paper to the computational method for solving the van der Pol equation. Readers interested in the modelling of the van der Pol equation can consult the literature [29]-[33]. The standard successive approximation method is able to

solve the problem only for temporal points close enough to the initial time, unless a large number of iterations are taken in the successive approximations [34]. However, taking a large number of successive iterations is impractical and should be avoided, because of the expensive computation. Therefore, an accurate method which is inexpensive computationally and simple practically is desired.

Our main contribution in this paper is to propose a new numerical-analytical method for solving the van der Pol equation. This contribution is useful, because the exact solution to the van der Pol equation is not available, at least until this paper is written. Our proposed method takes the strength of the variational iteration method, and at the same time, avoids the weakness of the variational iteration method. This leads that our proposed method is able to solve the van der Pol equation accurately for large time domain.

The rest of this paper is arranged as follows. We write the mathematical model for the van der Pol equation in section 2. Solving methods are presented in section 3. Results and discussion are provided in section 4. We conclude the paper in section 5.

2. MATHEMATICAL MODEL

The van der Pol equation is:

$$\frac{d^2 u(t)}{dt^2} - \mu(1 - u(t)^2) \frac{du(t)}{dt} + u(t) = 0, \quad (1)$$

with initial condition:

$$u(t_0) = u_0, \quad u'(t_0) = v_0, \quad (2)$$

where μ is a positive parameter indicating the strengths of the damping and the nonlinearity of the problem, t is the time variable, and $u(t)$ is the unknown function. The given domain is $t_0 \leq t \leq t_f$, where t_0 and t_f are the initial and the final of the time domain, respectively. Here $u'(t) = du/dt$. The van der Pol as shown in (1) describes self-excited oscillations in a triode electrical circuit. It was formulated by Balthasar van der Pol, a Dutch physicist, in around 1920 [29] for the electrical circuit problem, as shown in Figure 1. In this paper, we limit our research in the computational method for solving the van der Pol equation. Readers interested in the modelling of the van der Pol equation can consult the literature [29]-[33] and references therein.

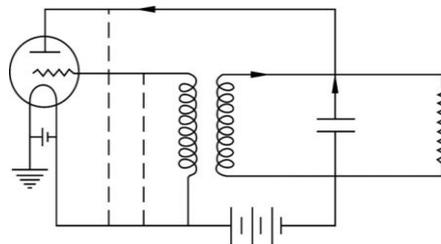


Figure 1. Illustration of the electrical circuit of van der Pol [29]

The van der Pol as shown in (1) can be written equivalently into the following system of two first order differential equations:

$$\frac{du}{dt} = v, \quad (3)$$

$$\frac{dv}{dt} = -u + \mu(1 - u^2)v. \quad (4)$$

Therefore, the solution to the van der Pol as shown in (1) is the solution to the system of as shown in (3) and (4). To obtain the solution to the van der Pol shown in (1), we shall solve the system of as shown in (3) and (4).

3. SOLVING METHODS

This section is devoted to present two solving methods for the considered van der Pol problem. We recall an existing iterative method. We also provide the proposed numerical-analytical iterative method. Our proposed method takes the strength and avoids the weakness of the existing iterative method.

3.1. Existing iterative method

An available iterative method in the literature is the successive approximation method. The successive approximation method for solving the van der Pol equation is described as follows. To begin, let us consider the general first order ordinary differential equation:

$$\frac{dy(x)}{dx} = f(x, y(x)), \quad (5)$$

with initial condition:

$$y(x_0) = y_0, \quad (6)$$

where the given domain is $x_0 \leq x \leq x_f$. Here x_0 is the initial point of the domain, x_f is the final point of the domain. Picard's successive approximation method for solving problem (5)-(6) is:

$$y_n(x) = y_0 + \int_{x_0}^x f(\xi, y_{n-1}(\xi)) d\xi, \quad (7)$$

where $n = 1, 2, 3, \dots$ and the exact solution $y(x)$ to problem (5)-(6) is given by the limit:

$$\lim_{n \rightarrow \infty} y_n(x) = y(x). \quad (8)$$

Following Picard's successive approximation method for solving problem (5)-(6), we develop a successive approximation method for solving the van der Pol as shown in (1) with initial condition (2). In this case, we shall solve the equivalent form (3)-(4) with initial condition (2). Taking the initialization:

$$u_0(t) = u_0, \quad v_0(t) = v_0, \quad (9)$$

and according to Picard's successive approximation method, we have the successive approximation method for problem (3)-(4) with initial condition (2) as shown in (10) and (11):

$$u_n(t) = u_0 + \int_{t_0}^t v_{n-1}(\tau) d\tau, \quad (10)$$

$$v_n(t) = v_0 + \int_{t_0}^t [-u_{n-1}(\tau) + \mu(1 - u_{n-1}(\tau)^2)v_{n-1}(\tau)] d\tau. \quad (11)$$

The successive approximation method is able to solve the van der Pol equation accurately only for a short time after the initialisation, unless we take a very large number of successive iterations. However, taking a very large number of successive iterations is very tedious and impractical. To deal with this problem, we propose a numerical-analytical method that we write in what follows.

3.2. Proposed numerical-analytical iterative method

We develop a numerical-analytical iterative method for solving the van der Pol equation. With initial condition (2) and for as shown in (3) and (4), let us take the following correction functionals:

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^t \lambda_1(\tau) \left[\frac{du_n(\tau)}{d\tau} - \bar{v}_n(\tau) \right] d\tau, \quad (12)$$

$$v_{n+1}(t) = v_n(t) + \int_{t_0}^t \lambda_2(\tau) \left[\frac{dv_n(\tau)}{d\tau} + \bar{u}_n(\tau) - \mu(1 - \bar{u}_n^2(\tau))\bar{v}_n(\tau) \right] d\tau, \quad (13)$$

where $\lambda_1(\tau)$ and $\lambda_2(\tau)$ are Lagrange multipliers to be determined optimally via the variational theory. Here $n = 0, 1, 2, \dots$. All variables having bar notations are restricted variables. Taking the variations of as shown in (12) and (13), we obtain:

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_{t_0}^t \lambda_1(\tau) \left[\frac{du_n(\tau)}{d\tau} \right] d\tau, \quad (14)$$

$$\delta v_{n+1}(t) = \delta v_n(t) + \delta \int_{t_0}^t \lambda_2(\tau) \left[\frac{dv_n(\tau)}{d\tau} \right] d\tau. \quad (15)$$

Rewriting shown in (14) and (15), we have:

$$\delta u_{n+1}(t) = \delta \left[(1 + \lambda_1(t))u_n(t) \right] - \delta \int_{t_0}^t \lambda'_1(\tau)u_n(\tau) d\tau, \quad (16)$$

$$\delta v_{n+1}(t) = \delta[(1 + \lambda_2(t))v_n(t)] - \delta \int_{t_0}^t \lambda'_2(\tau)v_n(\tau)d\tau. \quad (17)$$

We obtain stationary conditions based on shown in (16) and (17) as shown in (18) and (19):

$$1 + \lambda_1(t) = 0, \lambda'_1(\tau) = 0, \quad (18)$$

$$1 + \lambda_2(t) = 0, \lambda'_2(\tau) = 0. \quad (19)$$

The solutions to stationary conditions (18) and (19) are (20):

$$\lambda_1 = \lambda_2 = -1. \quad (20)$$

Therefore, the iterative formulas based on the taken correction functionals for solving the van der Pol equation are:

$$u_{n+1}(t) = u_n(t) - \int_{t_0}^t \left[\frac{du_n(\tau)}{d\tau} - v_n(\tau) \right] d\tau, \quad (21)$$

$$v_{n+1}(t) = v_n(t) - \int_{t_0}^t \left[\frac{dv_n(\tau)}{d\tau} + u_n(\tau) - \mu(1 - u_n^2(\tau))v_n(\tau) \right] d\tau. \quad (22)$$

Knowing the properties of the standard iterative method, we shall take advantage of the strength and avoid the weakness of the iterative method by implementing it into small sizes of the time intervals. To do so, the originally-given large size of the time domain is subdivided into small sizes of time intervals, and we implement the iterative method (21)-(21) into each of them. Considering the originally-given large size of time domain $t_0 \leq t \leq t_f$, we take discrete points $t_0, t_1, t_2, \dots, t_f$. These discrete points can be either equidistant or non-equidistant. For simplicity in this paper, we assume that we take $K + 1$ equidistant discrete points $t_0, t_1, t_2, \dots, t_K$, where $\Delta t = t_k - t_{k-1}$ for all k and $t_K = t_f$. In this way, we have K small same-size subintervals of time domain $I_k = [t_{k-1}, t_k]$ where $k = 1, 2, 3, \dots, K$. Suppose that we want to have N iterations in the numerical-analytical method for each subinterval. We denote $u_{n,k}(t)$ the approximate solution of $u(t)$ at the n th iteration on the k th subinterval. Notation $v_{n,k}(t)$ is meant analogously.

Therefore, the numerical-analytical method for solving the van der Pol equation works as follows. For $k = 1, 2, 3, \dots, K$ and for $n = 1, 2, 3, \dots, N$, we iterate:

$$u_{n,k}(t) = u_{n-1,k}(t) - \int_{t_{k-1}}^t \left[\frac{du_{n-1,k}(\tau)}{d\tau} - v_{n-1,k}(\tau) \right] d\tau, \quad (23)$$

$$v_{n,k}(t) = v_{n-1,k}(t) - \int_{t_0}^t \left[\frac{dv_{n-1,k}(\tau)}{d\tau} + u_{n-1,k}(\tau) - \mu(1 - u_{n-1,k}^2(\tau))v_{n-1,k}(\tau) \right] d\tau. \quad (24)$$

Here for the initialisation of the numerical-analytical method on each subinterval, if $k = 1$ we take:

$$u_{0,k}(t) = u_0, v_{0,k}(t) = v_0, \quad (25)$$

otherwise (if $k \geq 2$) we take:

$$u_{0,k}(t) = u_{N,k-1}(t_{k-1}), v_{0,k}(t) = v_{N,k-1}(t_{k-1}). \quad (26)$$

Computational tests will show that our proposed method is accurate on large sizes of time domain.

4. RESULTS AND DISCUSSION

In this section, we present our research results and discussion about them. First, we explain the numerical implementation of the proposed method. Then, two test cases are considered. The first test case takes $N = 5, t_0 = 0, t_f = 3, \mu = 0.1, u_0 = 1$, and $v_0 = 0$. The second test case takes the same parameter values as in the first one, except $\mu = 0.5$ and $u_0 = 2$. In this section, we use two abbreviations, namely, successive approximation method (SAM) and numerical-analytical method (NAM). SAM stands for the successive approximation method, which is the standard method. NAM stands for the numerical-analytical method, which is the one that we propose.

4.1. Numerical implementation of the proposed method

In the numerical implementation, the analytical iterative method must be written in the simplest possible form. This is in order that the computation is efficient. To simplify the numerical implementation, let us reconsider the iterative method (21)-(22), which can be written as shown in (27) and (28):

$$u_{n+1}(t) = u_n(t) - \int_{t_0}^t \left[\frac{du_n(\tau)}{d\tau} \right] d\tau + \int_{t_0}^t v_n(\tau) d\tau, \quad (27)$$

$$v_{n+1}(t) = v_n(t) - \int_{t_0}^t \left[\frac{dv_n(\tau)}{d\tau} \right] d\tau + \int_{t_0}^t [-u_n(\tau) + \mu(1 - u_n^2(\tau))v_n(\tau)] d\tau. \quad (28)$$

Then, we rewrite as shown in (27) and (28) as:

$$u_{n+1}(t) = u_0(t) + \int_{t_0}^t v_n(\tau) d\tau, \quad (29)$$

$$v_{n+1}(t) = v_0(t) + \int_{t_0}^t [-u_n(\tau) + \mu(1 - u_n^2(\tau))v_n(\tau)] d\tau. \quad (30)$$

This is interesting, because the iterative method (21)-(22) has become as shown in (29) and (30), which are actually another as shown in (10) and (11) with different indices.

Therefore, the numerical implementation of the proposed method for solving the van der Pol equation works as follows. For $k = 1, 2, 3, \dots, K$ and for $n = 1, 2, 3, \dots, N$, we iterate:

$$u_{n,k}(t) = u_{0,k}(t) + \int_{t_0}^t v_{n-1,k}(\tau) d\tau, \quad (31)$$

$$v_{n,k}(t) = v_{0,k}(t) + \int_{t_0}^t [-u_{n-1,k}(\tau) + \mu(1 - u_{n-1,k}^2(\tau))v_{n-1,k}(\tau)] d\tau. \quad (32)$$

Here for the initialisation of the numerical-analytical method on each subinterval, if $k = 1$ we take:

$$u_{0,k}(t) = u_0, \quad v_{0,k}(t) = v_0, \quad (33)$$

otherwise (if $k \geq 2$) we take:

$$u_{0,k}(t) = u_{N,k-1}(t_{k-1}), \quad v_{0,k}(t) = v_{N,k-1}(t_{k-1}). \quad (34)$$

4.2. Results and discussion for the existing method

We obtain that SAM solution is accurate only for small sizes of the time domain for both test cases, as shown in Figures 2 and 3 with $N = 5$. In Figure 2, we observe that SAM solution is inaccurate for t close to the final point of time. This phenomenon is even worse when the values of the nonlinearity parameter μ and the initial condition u_0 are higher, as we observe in Figure 3. We obtain that SAM is not applicable when the size of the time domain is large, because it is accurate only for temporal points about the initialisation.

4.3. Results and discussion for the proposed method

Here, for both test cases, NAM takes $\Delta t = 0.1$ and $N = 3$. To show that our proposed NAM method is accurate for large sizes of the time domain, we take the final time is large, say, $t_f = 50$. Reference and NAM solutions are shown in Figure 4 for the first test case and Figure 5 for the second test case, respectively. In these figures, we observe that NAM solutions provide close approximation to the reference solutions on the whole large domain. The curves of NAM solutions coincide graphically with the reference solutions, even though in NAM we have used smaller number of iterations ($N = 3$) than the one used in SAM, where in the previous SAM simulations we used five iterations ($N = 5$).

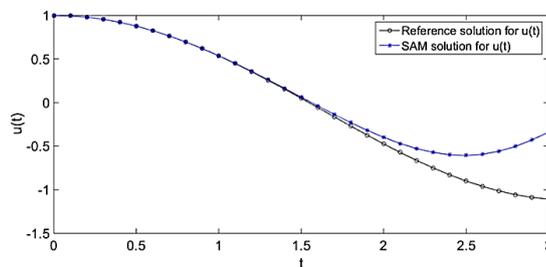


Figure 2. Reference and SAM solutions for the first test case of the van der Pol equation with $N = 5$, $t_0 = 0$, $t_f = 3$, $\mu = 0.1$, $u_0 = 1$, and $v_0 = 0$. SAM solution is inaccurate at points close to final time

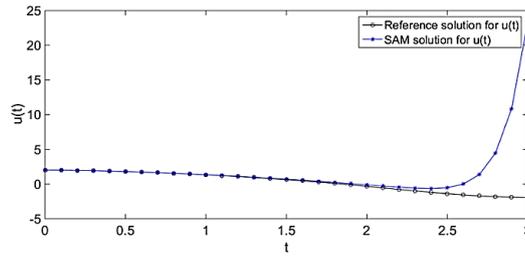


Figure 3. Reference and SAM solutions for the second test case of the van der Pol equation $N = 5$, $t_0 = 0$, $t_f = 3$, $\mu = 0.5$, $u_0 = 2$, and $v_0 = 0$. SAM solution is inaccurate at points close to final time

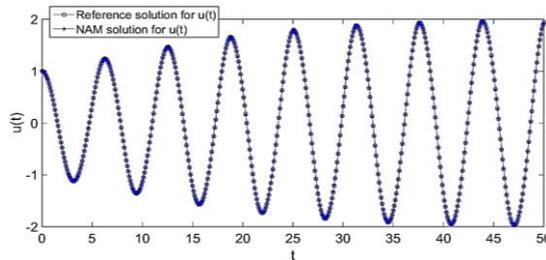


Figure 4. Reference and NAM solutions for the first test case of the van der Pol equation with $N = 3$, $t_0 = 0$, $t_f = 50$, $\mu = 0.1$, $u_0 = 1$, and $v_0 = 0$. NAM solution is accurate on large domain $[0, 50]$

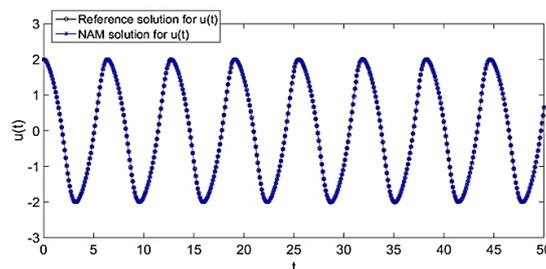


Figure 5. Reference and NAM solutions for the second test case of the van der Pol equation with $N = 3$, $t_0 = 0$, $t_f = 50$, $\mu = 0.5$, $u_0 = 2$, and $v_0 = 0$. NAM solution is accurate on large domain $[0, 50]$

4.4. Experimental convergence of the proposed method

For the first test, we record the absolute errors for variables $u(t)$ and $v(t)$ in Table 1. We observe that the experimental orders of convergence of our simulations are 3.05 and 3.03 for $u(t)$ and $v(t)$, respectively. This means that the experimental order of convergence of our simulations is 3 by rounding the numbers into the nearest integer. This matches with the number of iterations in the NAM evolution ($N = 3$).

For the second test case, we record the absolute errors for variables $u(t)$ and $v(t)$ in Table 2. We observe that the experimental orders of convergence of our simulations are 3.24 and 3.27 for $u(t)$ and $v(t)$, respectively. This means that again the experimental order of convergence of our simulations is 3, which is obtained by rounding the numbers into the nearest integer. Again, this matches with the number of iterations in the numerical-analytical evolution ($N = 3$).

Table 1. Average of absolute errors of NAM solutions for $u(t)$ and $v(t)$ for the first test case on $(0,50]$

Δt	Average of absolute errors for $u(t)$	Average of absolute errors for $v(t)$
0.1	5.083227E-04	4.899905E-04
0.05	6.137440E-05	5.994797E-05
Covergence rate	3.05	3.03

Table 2. Average of absolute errors of NAM solutions for $u(t)$ and $v(t)$ for the second test case on $(0, 50]$

Δt	Average of absolute errors for $u(t)$	Average of absolute errors for $v(t)$
0.1	8.813172E-04	8.801303E-04
0.05	9.319829E-05	9.136513E-05
Covergence rate	3.24	3.27

5. CONCLUSION

We have achieved our goal to propose a new numerical-analytical method for solving the van der Pol equation. As the exact solution to the van der Pol equation is not available, our proposed method is useful for obtaining approximate solutions simply and accurately. Solutions resulting from our proposed method are accurate and coincide graphically with available reference solutions for large sizes of the time domain. Future research direction could extend the proposed method for solving other engineering problems.

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