TELKOMNIKA Telecommunication, Computing, Electronics and Control
Vol. 20, No. 1, February 2022, pp. 9∼18
ISSN: 1693-6930, DOI: 10.12928/TELKOMNIKA.v20i1.19493

Radial radio number of chess board graph and king's graph

Kulandaivel Maruthamuthu Paramasivam[1], Kins Yenoke[2], Baby Smitha Kanaka Muralidharan[3]
[1]Department of Information Technology, University of Technology and Applied Sciences-Al Mussanah , Mussanah, Sultanate of Oman
[2]Department of Mathematics, Loyola College (Autonomous), Chennai, India
[3]Department of Mathematics, Devaswom Board College, Thalayolaparambu, Kottayam, India

Article Info

Article history:

Received Jan 13, 2021
Revised Dec 7, 2021
Accepted Dec 20, 2021

Keywords:

Chess board graphs
Ladder graph
King's graph
Radial radio labeling
Radial radio number

ABSTRACT

A radial radio labeling $\daleth$ of a connected graph $G = (V, E)$ with radius $rad(G)$ is a mapping from $V(G)$ to $N \cup \{0\}$ satisfying $|\daleth(u) - \daleth(w)| + d(u, w) \geq 1 + rad(G)$, $\forall\ u, v \in V(G)$. The span of a radial radio labeling $\daleth$, denoted by $rr(\daleth)$ is the greatest number in the range of $\daleth$. The minimum span taken over all radial radio labelings $\daleth$ of G is called the radial radio nmber of G and it is denoted by $rr(G)$. In this article, we have investigated the upper bounds for $rr(G)$ of chess board graphs and king's graph.

Corresponding Author:

Kulandaivel Maruthamuthu Paramasivam
Department of Information Technology, University of Technology and Applied Sciences-Al Mussanah
Mussanah, Sultanate of Oman
Email: mpkoman@gmail.com

1. INTRODUCTION

In today's digital age, our world has been transformed into a higher dimension by digital technology, especially in the field of communication technology. In the beginning of the 21^{st} century, Chartrand *et al.* [1] were motivated by the maximum channel allocation in a fixed spectrum introduced the concept of radio k-chromatic number in graph theory. For any k lies between 1 and the diameter of G, the radio $k-$ chromatic number is defined as follows: Let $G = (V, E)$ be connected graph and d be its diameter, then a radio k - coloring $\daleth$ of graph G is an assignment of $V(G)$ to non-negative integers such that $|\daleth(u) - \daleth(w)| + d(u, w) \geq 1 + k,\ \forall\ u, w \in V(G)$, where $d(u, w)$ is the distance between u and w in G. The span of a radio $k-$ chromatic number of h denoted by $r_{ck}(\daleth)$ is the largest number in the range of $\daleth$. The radio $k-$ chromatic number of G is the minimum value taken over all such radio k -chromatic number of $\daleth$. For different k values, different names were given to the radio $k-$ chromatic number by the researchers in the recent research articles. Namely, when the k values 1 and 2, then the problem is called the chromatic number and $L(2, 1)$ labeling number respectively. Chang and Kue [2] introduced the $L(2, 1)-$labeling of graphs. Hasunuma *et al.* [3] derived a linear time algorithm for $L(2, 1)$ labeling of trees. Yenoke *et al.* [4], [5] determined the upper bound for the $L(2, 1)$ problem of slim tree, comb graph, double comb graph and Christmas tree, silicate and oxide networks as 8, 6, 7, 8, 10 and 8 respectively. Prajapati and Patel [6] obtained the $L(2, 1)$ labeling number of crown graph and line graph of armed crown graph. Smitha and Thirusangu [7] determined the $L(2,1)$ labeling of cycle related graphs. Besides, $L(2,1)$ labeling of unigraph was computed by Calamoneri and Petreschi [8].

If we vary the value of k as 3, then the problem is named as $L(3,2,1)$ labelling. Xavier *et al.* [9] obtained the bounds for the $L(3,2,1)$ labelling number of for both n-star graph $S(n,r)$ and n-star-wheel graph $SW(n,r)$ as $2r+2n+1$ and n-wheel star graph $WS(n,r)$ as $3r+2n$. Kim *et al.* [10] solved the $L(3,2,1)$ labeling problem the product of C_n and K_n. Moreover, Amanathulla and Pal [11] studied the same on interval graphs. Further, when k reaches the maximum value d, this problem insistence the assigning of channels to FM radio stations called radio labeling problem which was introduced by Chartrand [12]. Rajan *et al.* [13] and Rajan and Yenoke [14] obtained the lower bound for the radio number of any connected graph and also investigated the exact radio number of wheel graph W_{n+1}, double fan graph DF_n, fan graph F_n, windmill graph K_n^m, star graph S_{n+1} and uniform r -cyclic split graph $KC(r)$ as $n+2, n+3, n+2, m(n-1)+2$, $n+3$ and $nkr+3n-2$ respectively.

Vaidya and Bantva [15] attained the exact radio number of the total graph of path P_n for $n=2k$ and $n=2k+1$ as $4k(k-1)+2$ and $4k^2+3$ respectively. Recently, Yenoke and Kaabar [16] investigated the bounds for the Nanostar tree dendrimer $T_{n,p}(n,p>2)$ as $rn(T_{n,p}) \leq n+(2n-1)p+1+\Sigma_{l=1}^{n-1}(2l-1)p((p-1)^{n-l-1})(p-2)+(2l-1)p(p)^{n-l-1}-1$, whenever $p \geq 2n-3$ and $p(\frac{(p-1)^n-1}{p-2})+n+(\Sigma_{i=1}^{n-1}2(2n-(2n-i))p(p-1)^{n-i-1})$ respectively. In addition, when $k=d-1$, it was named as antipodal radio number by [17], [18]. William and Kenneth [19] investigated the bounds for the antipodal radio number of lobster graph as $an(L(m,r,k)) \leq 24r+20k-7$. Saha and Panigrahi [20] studied the same problem for some powers of cycles. Avadayappan *et al.* [21] were introduced the radial radio labeling by fixing the k value as the radius of the graph.

A radial radio labeling $\urcorner$ of a connected graph $G=(V,E)$ with radius $rad(G)$ is a mapping from $V(G)$ to $N \cup \{0\}$ satisfying $|\urcorner(u)-\urcorner(w)| + d(u,w) \geq 1+rad(G)$, $\forall\ u,v \in V(G)$. The span of a radial radio labeling $\urcorner$, denoted by $rr(\urcorner)$ is the greatest number in the range of $\urcorner$. The minimum span taken over all radial radio labelings $\urcorner$ of G is called the radial radio nmber of G and it is denoted by $rr(G)$. This problem is very helpful in dividing a network into sub networks and to apply the radio labeling conditions in assigning the channels for a particular divided geographical area. Especially, if there is a need of partitioning the existing network into two sub networks, this labeling technique can be applied without affecting the optimal channel assignment.

Hence this method is either used to increase the number of channels or can be used for the same frequency allocation in different geographical area. Since this problem was recently introduced in 2019, only few research articles were published. Avadayappan *et al.* [21], [22] were proved the following significant results: (i) For any simple connected graph G, $rr(G) \geq \omega(G)$. (ii) For any graph G with $m \geq 1$, there is a graph G with $\omega=3$ and $rr(G)=m+\omega$ (iii) For any graph G with $\omega \geq 4$, there exists a graph G with $rr(G)=\omega+1$. (iv) For any self centered graph $rr(G) \geq n$, where $n=V(G)$. Yenoke [23] investigated $rr(G)$ of certain uniform cyclic and wheel split graphs as $rr(KDW(r)) \leq 3(r+2n), n>4, rr(HW(r)) \leq 2(r+n)+2, n>3$ rr$(SW(r)) \leq 2r+3n, n>1, r(KC(r))) = mr+2n-2, m>1$ and $rr(KW(r)) \leq 2r+4(n-1), n>1$.

Recently, Jose and Giridharan [24] proved that $rr(MT(n)) \leq 2n+1$ and $rr(D(n)) \leq 2n+2$, where $MT(n)$ and $D(n)$ are Mongolian tent and diamond graphs respectively. In this paper we have estimated the bounds for the radial radio number of certain interconnection networks such as chess board graphs and King's graph.

2. DEFINITIONS AND TERMINOLOGY

In this section we have listed few definitions and results which will be used for proving the theorems. Let u be a vertex of a connected graph G, then the eccentricity of u denoted $e(u)$ is the farthest vertex from u to any other vertex v in G. That is, $e(v)=\max\{d(u,v)\forall v \in V(G)\}$. The radius of the graph G is the minimum eccentricity of the vertices of G and it is denoted by $\mathrm{rad}(G)$. Pardalos *et al.* [25] defined the following chess board and its related graphs.

An $m \times n$ chessboard graph denoted by $\mathrm{CB}(\mathrm{m}, n)$ is defined as the Cartesian product $P_m \times P_n$ of paths on m and n vertices respectively. In the literature it is also denoted by $m \times n$ mesh. The mn vertices in $CB(m, n)$ are named as $\{(k, l) \backslash l = 1, 2 \ldots m, k = 1, 2 \ldots n\}$. If $m = n$, then the radius of $CB(m, n)$ is $2\left\lfloor \frac{n}{2} \right\rfloor$. A $2 \times n$ chessboard graph with $2n$ vertices is also called a ladder graph denoted by L_n. The radius of L_n is $\left\lfloor \frac{n}{2} \right\rfloor + 1$. An $m \times n$ King's graph denoted by $KG(m, n)$ is a graph which is obtained by all legal moves of the king chess piece on a $m \times n$ chessboard $CB(m, n)$. More specifically, it is constructed by the strong product of the paths P_m and P_n. The radius of $KG(m, n)$ is $\left\lfloor \frac{n}{2} \right\rfloor$.

3. MAIN RESULTS

In this section we have obtained the bounds the radial radio number of $2 \times n$ and $n\times\, n$ chessboard graph separately. Further, we have determined the bounds for the radial radio number of $n \times n$ king's graph.

a) Theorem 1: Let L_n be a ladder graph with $2n$ vertices, then the radial radio number of L_n satisfies

$$rr\left(L_n\right) \leq \begin{cases} \frac{n(n+2)}{4} + 1, & \text{if } n \text{ is even} \\ \frac{\left(n^2-1\right)}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Proof: We prove this theorem using two cases based on the value of n, odd or even. Define a radial radio labeling from the vertex set of L_n to the non-negative integers separately for odd and even cases as follows:

1) Case 1: n is even.

Define a labeling from the vertex set of L_n to the non-negative integers as follows:

$\daleth(0, i) = 2(i-1)r - (i-1),\ i = 1, 2 \ldots \frac{n}{2} + 1$

$\daleth(1, i) = (2i-1)r - (i-1),\ i = 1, 2 \ldots \frac{n}{2} + 1$

$\daleth\left(0, \frac{n}{2} + i\right) = (2i-1)r - (i-1),\ i = 1, 2 \ldots \frac{n}{2} - 1$

$\daleth\left(1, \frac{n}{2} + i\right) = 2(i-1)r - (i-1),\ i = 1, 2 \ldots \frac{n}{2} - 1.$

See the Figure 1.

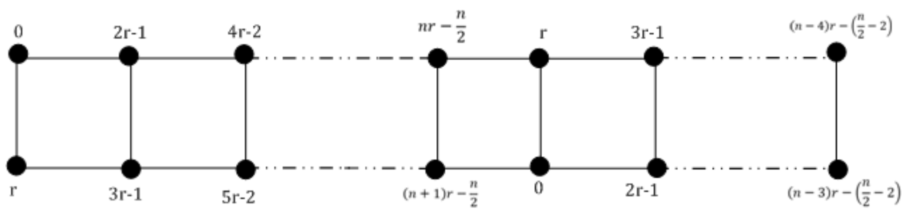


Figure 1. A radial radio labeling of ladder graph L_n (n even) which attains the bound

Claim: $\daleth$ is a valid radial radio labeling.

Since n is even, the radius of L_n is $\frac{n}{2}+1$. Therefore, we must prove that $|\daleth(u) - \daleth(w)| + d(u, w) \geq \frac{n}{2} + 2$ for all $u, w \in V\left(L_n\right)$

1.1) Case 1.1: Suppose $u \neq w$ belongs to the upper row of L_n, then the following three sub cases can arise.

- Case 1.1.1: If $u = (0, j)$ and $w = (0, k), 1 \leq j \neq k \leq \frac{n}{2} + 1$, then $7(u) = 2(j-1)r - (j-1)$ and $\daleth(w) = 2(k-1)r - (k-1)$. Also, $d(u, w) \geq 1$.
  Hence, $|\daleth(u) - \daleth(w)| + d(u, w) \geq 1 + |2(j-k)r| \geq 2\frac{n}{2} + 1 \geq \frac{n}{2} + 2$, since $j \neq k$ and $r = \frac{n}{2} + 1$

- Case 1.1.2: If $u = (0, j)$ and $w = (0, k), \frac{n}{2} + 2 \leq j \neq k \leq n$, then $7(u) = (2j-1)r - (j-1)$ and $\daleth(w) = (2k-1)r - (k-1)$. Also, $d(u, w) \geq 1$ and $|\daleth(u) - \daleth(w)| \geq| (2j-1)\, r - (2k-1)r| \geq |2r\ |$, since $j \neq k$. Hence $|\daleth(u) - \daleth(w)| + d(u, w) \geq 1 + 2\left(\frac{n}{2} + 1\right) > \frac{n}{2} + 2$
- Case 1.1.3: If $u = (0, j), 1 \leq j \leq \frac{n}{2} + 2$ and $w = (0, k), \frac{n}{2} + 2 \leq k \leq n$, then $\daleth(u) = 2(j-1)r - (j-1)$ and $\daleth(w) = (2k-1)r - (k-1)$. Here $d(u, w)$ ranges from 1 to $n - 1$. Hence the required condition becomes, $|\daleth(u) - \daleth(w)| + d(u, w) \geq 1 + |2(j-1)r - (j-1) - ((2k-1)r - (k-1))| \geq 1 + r = \frac{n}{2} + 2$

1.2) Case 1.2: Suppose $u \neq w$ belongs to the lower row of L_n, then we will arrive the same three subcases as in Case 1.1, because the labeling in the first half of the upper row is same as the second half of the lower row and vice versa.

1.3) Case 1.3: Let u be a vertex on the upper row and w be a vertex on the lower row.

- Case 1.3.1: If $u = (0, j)$ and $w = (1, k), 1 \leq j, k \leq \frac{n}{2} + 1$, then $7(u) = 2(j-1)r - (j-1)$ and $\daleth(w) = (2k-1)r - (k-1)$. Since the function values are same, as in Case 1.1.3, $|\daleth(u) - \daleth(w)| + d(u, w) \geq \frac{n}{2} + 2$.
- Case 1.3.2: Suppose $u = (0, j), 1 \leq j \leq \frac{n}{2} + 1$ and $w = (1, k), \frac{n}{2} + 2 \leq k \leq n$, then $\daleth(u) = 2(j-1)r - (j-1)$ and $\daleth(w) = 2(k-1)r - (k-1)$.
  If $j = k$, then the distance between them is exactly $\frac{n}{2} + 2$, and hence the condition is proved. If $j \neq k$, then $|\daleth(u) - \daleth(w)| \geq |2(j-1)r - (j-1) - (2(k-1)r - (k-1))| \geq| 2(j-k\ )\, r + (j-k)\ |$ and $d(u, w) \geq 1$. Therefore, $|\daleth(u) - \daleth(w)| + d(u, w) \geq \frac{n}{2} + 2$, since $j \neq k$. The rest of the sub cases also follows the similar proof as that of the previous ones. Hence $\daleth$ is a valid radial radio labeling. Also, $\daleth$ received the maximum value $\frac{n(n+2)}{4} + 1$ at the vertex $\left(1, \frac{n}{2} + 1\right)$. That is, $\daleth\left(1, \frac{n}{2} + 1\right) = \left(2\left(\frac{n}{2} + 1\right) - 1\right) r - \left(\frac{n}{2} + 1 - 1\right) = (n+1)\left(\frac{n}{2} + 1\right) - \frac{n}{2} = \frac{n(n+2)}{4} + 1$. Therefore $rr\,(L_n) \leq \frac{n(n+2)}{4} + 1$, when n is even.

2) Case 2: n is odd

Define a mapping $\daleth : V\,(L_n) \to N \cup \{0\}$ as follows: $\daleth(0, i) = (2r-1)(i-1),\ i = 1, 2 \ldots \frac{+1}{2}$
$\daleth(1, i) = (2i-1)r - (i-1),\ i = 1, 2 \ldots \frac{n+1}{2}$
$\daleth\left(0, \frac{n}{2} + i\right) = (2i-1)r - (i-1),\ i = 1, 2 \ldots \frac{n-1}{2}$
$\daleth\left(1, \frac{n}{2} + i\right) = (2r-1)(i-1),\ i = 1, 2 \ldots \frac{n-1}{2}$

Figure 2 illustrates the proof of this case. Hence $rr\,(L_n) \leq \begin{cases} \frac{n(n+2)}{4} + 1, & n \text{ even} \\ \frac{(n^2-1)}{2}, & n \text{ odd} \end{cases}$

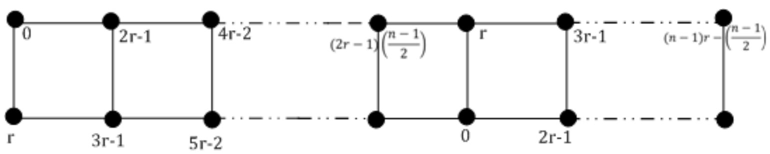


Figure 2. A ladder graph L_n (n odd) and its mapping $\daleth$

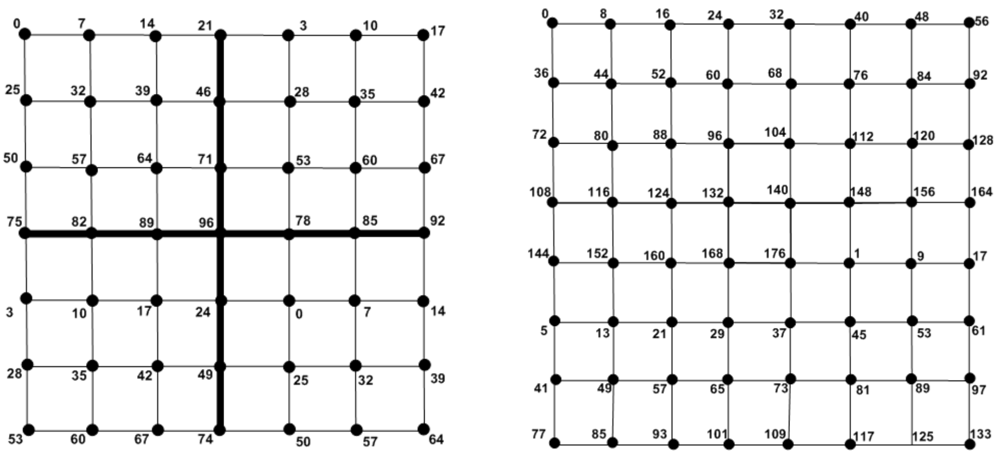
b) Theorem 2: Let $n > 2$ be odd, then the radial radio number of $n \times n$ chessboard graph $CB(n, n)$ satisfies $rr(CB(n, n)) \leq \frac{(n^2-1)(n+1)}{4}$.

Proof: First we partition the vertex set $V(CB(n, n)) = \{(i, j), j = 1, 2 \ldots n, i = 1, 2 \ldots n\}$ into 4 disjoint sets V_1, V_2, V_3 and V_4 as given below:

$V_1 = \left\{(i,j) : j = 1,2\ldots\frac{n+1}{2},\ i = 1,2\ldots\frac{n+1}{2}\right\}$,
$V_2 = \left\{\left(i, \frac{n+1}{2}+j\right) : j = 1,2\ldots\frac{n-1}{2},\ i = 1,2\ldots\frac{n+1}{2}\right\}$
$V_3 = \left\{\left(\frac{n+1}{2}+i, j\right) : j = 1,2\ldots\frac{n+1}{2},\ i = 1,2\ldots\frac{n-1}{2}\right\}$,
$V_4 = \left\{\left(\frac{n+1}{2}+i, \frac{n+1}{2}+j\right) : j = 1,2\ldots\frac{n-1}{2},\ i = 1,2\ldots\frac{n-1}{2}\right\}$. Clearly $V_i \cap V_j = \emptyset$. Next, we define a mapping $\daleth : V(CB(n,n)) \to N \cup \{0\}$ as follows:
$\daleth((i,j)) = \frac{(n^2+1)}{2}(i-1) + n(j-1),\ j = 1,2\ldots\frac{n+1}{2},\ i = 1,2\ldots\frac{n+1}{2}$
$\daleth\left(\left(i, \frac{n+1}{2}+j,\right)\right) = n(j-1) + \left(\frac{n^2+1}{2}\right)(i-1) + \frac{n-1}{2},\ j = 1,2\ldots\frac{n-1}{2},\ i = 1,2\ldots\frac{n+1}{2}$
$\daleth\left(\left(\frac{n+1}{2}+i, j\right)\right) = n(j-1) + \left(\frac{n^2+1}{2}\right)(i-1) + \frac{n-1}{2},\ j = 1,2\ldots\frac{n+1}{2},\ i = 1,2\ldots\frac{n-1}{2}$
$\daleth\left(\left(\frac{n+1}{2}+i, \frac{n+1}{2}+j\right)\right) = \frac{(n^2+1)}{2}(i-1) + n(j-1),\ j = 1,2\ldots\frac{n-1}{2},\ i = 1,2\ldots\frac{n-1}{2}$.
See Figure 3(a).

Now we claim that, the radial radio labeling condition is true for any pair of vertices in $CB(n,n)$. since n is odd, the radius of $M(n,n)$ is $n-1$. Hence, we must show that $|\daleth(u) - \daleth(w)| + d(u,w) \geq n \forall u,w \in V(CB(n,n))$. Let u and w be any two vertices in the chess board graph $CB(n,n)$. Suppose u and w lie in any one of the mutually disjoint sets, then u and w are labelled with a difference at least n That is, $|\daleth(u) - \daleth(w)| \geq n$. Therefore, there is nothing to verify the radial radio labeling condition in these cases. Hence, we proceed to check the remaining cases.

1) Case 1: Let $u \in V_1$ and $w \in V_2$, then $u = (l,k)$ and $w = \left(s, \frac{n+1}{2}+t\right)$, $1 \leq l,k,s \leq \frac{n+1}{2}, 1 \leq t \leq \frac{n-1}{2}$. Therefore, the corresponding labelings of u and w are $\frac{(n^2+1)}{2}(l-1) + n(k-1)$ and $n(t-1) + \left(\frac{n^2+1}{2}\right)(s-1) + \frac{n-1}{2}$ respectively.
   1.1) Case 1.1: If $l = s$ and $k = t$, then the distance between them is exactly $\frac{n+1}{2}$. Therefore $|\daleth(u) - \daleth(w)| + d(u,w) \geq \frac{n+1}{2} + \left|\left(\frac{(n^2+1)}{2}(l-1) + n(k-1)\right) - (n(t-1) + \left(\frac{n^2+1}{2}\right)(s-1) + \left(\frac{n-1}{2}\right)\right)\right| = \frac{n+1}{2} + \frac{n-1}{2} = n$.
   1.2) Case 1.2: If $l = s$ and $k = t+1$, then $d(u,w) = \frac{n-1}{2}$ and $|\daleth(u) - \daleth(w)| = \left|\left(\frac{(n^2+1)}{2}(l-1) + n(t+1-1)\right) - \left(n(t-1) + \left(\frac{n^2+1}{2}\right)(s-1) + \left(\frac{n-1}{2}\right)\right)\right| \geq \frac{n+1}{2}$. Therefore $|\daleth(u) - \daleth(w)| + d(u,w) \geq \frac{n-1}{2} + \frac{n-1}{2} = n$. Also, for the remaining possibilities in this case, the modulus difference of $\daleth(u)$ and $\daleth(w)$ is at least $n + \left(\frac{n-1}{2}\right) > n$.



(a) (b)

Figure 3. A radial radio labeling of (a) CM(7,7) and (b) CM(8,8) which illustrates the mapping

2) Case 2: Let $u \in V_1$ and $w \in V_3$, then $u = (l,k)$ and $w = \left(\frac{n+1}{2} + s, t\right), 1 \leq l,k,t \leq \frac{n+1}{2}, 1 \leq s \leq \frac{n-1}{2}$. Therefore, $\daleth(u) = \frac{(n^2+1)}{2}(l-1) + n(k-1)$ and $\daleth(w) = n(t-1) + \left(\frac{n^2+1}{2}\right)(s-1) + \frac{n-1}{2}$ respectively. The rest of the proof in this case is similar to Case 1.

3) Case 3: Suppose $u \in V_1$ and $w \in V_4$, then $\daleth(u) = \frac{(n^2+1)}{2}(l-1) + n(k-1), 1 \leq l,k, \leq \frac{n+1}{2}$ and $\daleth(w) = \frac{(n^2+1)}{2}(s-1) + n(t-1), 1 \leq s,t \leq \frac{n-1}{2}$, where $u = (l,k)$ and $w = \left(\frac{n+1}{2} + s, \frac{n+1}{2} + t\right)$.
   3.1) Case 3.1: If $l = s$, then $d(u,w) \geq 1$. Hence the radial radio labeling condition becomes, $|\daleth(u) - \daleth(w)| + d(u,w) > n$.
   3.2) Case 3.2: If $s = l+1$, then $d(u,w) = n-2$ and $|\daleth(u) - \daleth(w)| \geq \frac{n+1}{2}$. Therefore $|\daleth(u) - \daleth(w)| + d(u,w) > n$.
   3.3) Case 3.3: If $l \neq s$ and $s \neq l+1$, then $d(u,w) \geq 2$ and $|\daleth(u) - \daleth(w)| \geq n$. Hence $|\daleth(u) - \daleth(w)| + d(u,w) > n$.

4) Case 4: Let $u \in V_2$ and $w \in V_3$, then $u = \left(l, \frac{n+1}{2} + k\right)$ and $w = \left(\frac{n+1}{2} + s, t\right), 1 \leq l,t \leq \frac{n+1}{2}$ $1 \leq k,s \leq \frac{n-1}{2}$. From the mapping, the values of $\daleth(u)$ and $\daleth(w)$ are $n(k-1) + \left(\frac{n^2+1}{2}\right)(l-1) + \frac{n-1}{2}$ and $n(t-1) + \left(\frac{n^2+1}{2}\right)(s-1) + \frac{n-1}{2}$ respectively.

   If we proceed in the same way as in Case 3, we can easily verify the radial radio labeling condition satisfies for this case also. The remaining cases namely $u \in V_2, w \in V_4$ is similar to case 2 and $u \in V_2, w \in V_4$ is similar to case 1. Thus, we have verified that $|\daleth(u) - \daleth(w)| \geq n \forall u,w \in V(CB(n,n))$. Also, the maximum value of h attains at the vertex $\left(\frac{n+1}{2}, \frac{n+1}{2}\right)$. Hence, we substitute the values of i and j as $\frac{n+1}{2}$, we get $rr(\daleth) = \daleth\left(\frac{n+1}{2}, \frac{n+1}{2}\right) = \frac{(n^2+1)}{2}\left(\frac{n+1}{2} - 1\right) + n\left(\frac{n+1}{2} - 1\right) = \left(\frac{n+1}{2} - 1\right)\frac{(n+1)^2}{2} = \frac{(n^2-1)(n+1)}{4}$ Thus, $rr(G) \leq \frac{(n^2-1)(n+1)}{4}$.

c) Theorem 3: For $n > 2$, the radial radio number of $n \times n$ chessboard graph $CB(n,n)$ satisfies $rr(CB(n,n)) \leq \frac{n^2(n+3)}{4}$, whenever n is even.

   Proof: Define $\daleth : V(CB(n,n)) \to N \cup \{0\}$ as follows:
   $\daleth((i,j)) = \frac{n(n+1)}{2}(i-1) + n(j-1),\ j = 1,2\ldots n,\ i = 1,2\ldots\frac{n}{2}$
   $\daleth\left(\left(\frac{n}{2} + 1 + i, j\right)\right) = \frac{n(n+1)}{2}(i-1) + n(j-1) + \frac{n}{2} + 1,\ j = 1,2\ldots n,\ i = 1,2\ldots\frac{n}{2} - 1$
   $\daleth\left(\left(\frac{n}{2} + 1, j\right)\right) = n(j-1) + \frac{n^2(n+1)}{4},\ j = 1,2\ldots\frac{n}{2} + 1$
   $\daleth\left(\left(\frac{n}{2} + 1, \frac{n+1}{2} + j\right)\right) = n(j-1) + 1,\ j = 1,2\ldots\frac{n}{2} - 1$. See Figure $3(b)$.

   Since n is even, the radius of $CB(n,n)$ is n. Hence as in Theorem 3, we can easily verify that $|\daleth(u) - \daleth(w)| + d(u,w) \geq n+1$ holds for every pair of vertices $u,w \in V(CB(n,n))$ Also, the vertex $\left(\frac{n}{2} + 1, \frac{n}{2} + 1\right)$ attains the maximum value under the mapping $\daleth$. Therefore, $\daleth\left(\left(\frac{n}{2} + 1, \frac{n}{2} + 1\right)\right) = n\left(\frac{n}{2} + 1 - 1\right) + \frac{n^2(n+1)}{4} = \frac{n^2(n+3)}{4}$. Hence $rr(CBM(n,n)) \leq \frac{n^2(n+3)}{4}$, whenever n is even.

d) Theorem 4: Let n be even. Then for the king's graph $KG(n,n)$ satisfies $rr(KG(n,n)) \leq \frac{n^2}{4} + \left(\frac{n}{2}\right)\left((n-1)\left(\frac{n}{2} - 1\right) + 2\right) - 2, n > 2$.

   Proof: Define a mapping $\daleth$: V$(KG(n,n)) \to N \cup \{0\}$ as follows:
   $\daleth(i,1) = (i-1)\left(\frac{n}{2}\right),\ i = 1,2,\ldots,\frac{n}{2} + 1,$
   $\daleth(\frac{n}{2} + 1 + i, 1) = (i-1)\left(\frac{n}{2}\right),\ i = 1,2,\ldots,\frac{n}{2} - 1,$
   $\daleth(i, \frac{n}{2} + 1) = i\left(\frac{n}{2}\right) - 1,\ i = 1,2,\ldots,\frac{n}{2} + 1,$
   $\daleth(\frac{n}{2} + 1 + i, \frac{n}{2} + 1) = i\left(\frac{n}{2}\right) - 1,\ i = 1,2,\ldots,\frac{n}{2} - 1,$
   $\daleth(i, j+1) = \frac{n^2}{4} + \left(\frac{n}{2}\right)((i-1) + (n-1)(j-1)) + j,\ i = 1,2,\ldots,n,\ j = 1,2,\ldots,\frac{n}{2} - 1,$
   $\daleth(i, \frac{n}{2} + j + 1) = \frac{n^2}{4} + \left(\frac{n}{2}\right)((i-1) + (n-1)(j-1) + 1) + j - 1,\ i = 1,2,\ldots,n,\ j = 1,2,\ldots,\frac{n}{2} - 1.$
   See the Figure $4(a)$. As the radius of $KG(n,n)$ is $\frac{n}{2}$, we must verify the radial radio labeling condition $|\daleth(u) - \daleth(w)| + d(u,w) \geq 1 + \frac{n}{2} \forall u,w \in V(KG(n,n))$

1) Case 1: Suppose that u and w are of the form $\left(s, \frac{n}{2}+1\right)$ and $\left(t, \frac{n}{2}+1\right)$ where $1 \leq s \neq t \leq \frac{n}{2}+1$. Then $\urcorner\left(s, \frac{n}{2}+1\right)=\left(\frac{n}{2}\right) s, \urcorner\left(t, \frac{n}{2}+1\right)=\left(\frac{n}{2}\right) t$ and $d(u, w)>0$. Consequently $|\urcorner(u)-\urcorner(w)|+d(u, w) \geq\left|\left(\frac{n}{2}\right)(s-t)\right|+1 \geq \frac{n}{2}+1$
2) Case 2: Let us take $u=\left(\frac{n}{2}+s, \frac{n}{2}+1\right)$ and $w=\left(\frac{n}{2}+t, \frac{n}{2}+1\right)$, then $\urcorner(u)=\left(\frac{n}{2}\right) s$ and $\urcorner(w)=\left(\frac{n}{2}\right) t, 1 \leq s \neq t \leq \frac{n}{2}+1$. As $d(u, w) \geq 1,|\urcorner(u)-\urcorner(w)|+d(u, w) \geq \frac{n}{2}+1$.
3) Case 3: Suppose $u=(k, l+1)$ and $w=(s, t+1), 1 \leq k, s \leq n, 1 \leq l, t \leq \frac{n}{2}-1$, then $|\urcorner(u)-\urcorner(w)|=\left|\left(\frac{n^2}{4}+\left(\frac{n}{2}\right)((k-1)+(n-1)(l-1))+l\right)-\left(\frac{n^2}{4}+\left(\frac{n}{2}\right)((s-1)+(n-1)(t-1))+t\right)\right|$.
   - 3.1) Case 3.1: Allowing $k \neq s$, we get $d(u, w) \geq 1$ and $|\urcorner(u)-\urcorner(w)| \geq \frac{n}{2}$, which confirm the result.
   - 3.2) Case 3.2: If $l \neq t$, then $|\urcorner(u)-\urcorner(w)| \geq\left(\frac{n}{2}\right)(n-1)$ and $d(u, w)>0$ which verifies $|\urcorner(u)-\urcorner(w)|+d(u, w) \geq \frac{n}{2}+1$, since $n>2$.
4) Case 4: Let $u=\left(k, \frac{n}{2}+l+1\right)$ and $w=\left(s, \frac{n}{2}+t+1\right), 1 \leq k, s \leq n, 1 \leq l, t \leq \frac{n}{2}-1$ If $k \neq s$, then $|\urcorner(u)-\urcorner(w)|=\left|\left(\frac{n^2}{4}+\left(\frac{n}{2}\right)((k-1)+(n-1)(l-1)+1)+l-1\right)-\left(\frac{n^2}{4}+\left(\frac{n}{2}\right)((s-1)+(n-1)(t-1)+1)+t-1\right)\right| \geq \frac{n}{2}$ Again, $l \neq t$ implies that $|\urcorner(u)-\urcorner(w)| \geq\left(\frac{n}{2}\right)(n-1)$. Since $d(u, w) \geq 1$, whence in both the possibilities, the condition $|\urcorner(u)-\urcorner(w)|+d(u, w) \geq \frac{n}{2}+1$ is verified.
5) Case 5: Suppose $u=(k, 1)$ of $\left(\frac{n}{2}+1+s, 1\right)$ d $u=(l, 1)$ or $\left(\frac{n}{2}+1+t, 1\right)$, then either $|\urcorner(u)-\urcorner(w)|=0$ and $d(u, w) \geq \frac{n}{2}+1$ or $|\urcorner(u)-\urcorner(w)|=\frac{n}{2}$ and $d(u, w) \geq 1$. Hence in both chances $|\urcorner(u)-\urcorner(w)|+d(u, w) \geq \frac{n}{2}+1$.
6) Case 6: Assume $u=(i, 1)$ and $w=\left(i, \frac{n}{2}+1\right), 1 \leq i \leq \frac{n}{2}+1$, then $\urcorner(u)=(i-1)\left(\frac{n}{2}\right)$ $\urcorner(w)=i\left(\frac{n}{2}\right)-1$ and $d(u, w) \geq 1$. Therefore, $|\urcorner(u)-\urcorner(w)|+d(u, w) \geq \frac{n}{2}+1$.
7) Case 7: If $u=(i, j+1)$ and $w=\left(k, \frac{n}{2}+l+1\right)$, then $\urcorner(u)=i\left(\frac{n}{2}\right)-1$ and $\urcorner(w)=\frac{n^2}{4}+\left(\frac{n}{2}\right)((k-1)+(n-1)(l-1)+1)+l-1, \quad 1 \leq i, l \leq \frac{n}{2}-1,1 \leq j, k \leq n$. Therefore, $|\urcorner(u)-\urcorner(w)|+d(u, w)>\frac{n}{2}+1$.

   Similarly, we can verify the radial radio condition for the rest of the cases. Thus, $\urcorner$ is a valid radial radio labeling and which attains the maximum label $\frac{n^2}{4}+\left(\frac{n}{2}\right)\left((n-1)+(n-1)\left(\frac{n}{2}-2\right)+1\right)+\frac{n}{2}-1=\frac{n^2}{4}+\left(\frac{n}{2}\right)\left((n-1)\left(\frac{n}{2}-1\right)+2\right)-2$ for the vertex (n, n). Thus $rr(KG(n, n)) \leq n^2/4+(n/2)((n-1)(n/2-1)+2)-2, n>2$.

e) Theorem 5: Let n be odd. Then the radial radio number of $KG(n, n)$ satisfies $rr(KG(n, n)) \leq \left\lfloor\frac{n}{2}\right\rfloor\left(\left\lceil\frac{n}{2}\right\rceil-1\right)+\left\lfloor\frac{n}{2}\right\rfloor\left((n-1)\left\lfloor\frac{n}{2}\right\rfloor+1\right), n>2$.

Proof: Define a mapping $\urcorner: V(KG(n, n)) \rightarrow N \cup\{0\}$ as follows:

$\urcorner(i, 1)=(i-1)\lfloor\frac{n}{2}\rfloor,\ i=1,2, \ldots,\lceil\frac{n}{2}\rceil$,

$\urcorner\left(\lfloor\frac{n}{2}\rfloor+i, 1\right)=(i-1)\lfloor\frac{n}{2}\rfloor,\ i=1,2, \ldots,\lfloor\frac{n}{2}\rfloor$,

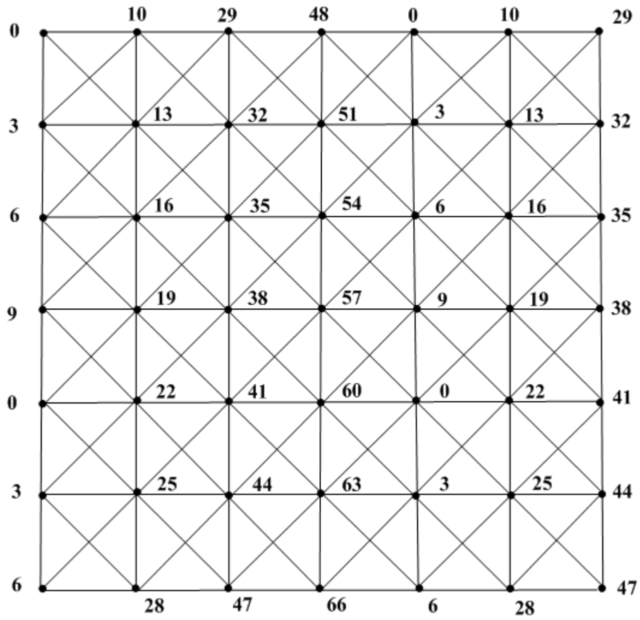
$\urcorner\left(i,\lceil\frac{n}{2}\rceil+1\right)=i\lfloor\frac{n}{2}\rfloor-1,\ i=1,2, \ldots,\lceil\frac{n}{2}\rceil$,

$\urcorner\left(\lceil\frac{n}{2}\rceil+i,\lceil\frac{n}{2}\rceil+1\right)=i\lfloor\frac{n}{2}\rfloor-1,\ i=1,2, \ldots,\lfloor\frac{n}{2}\rfloor$,

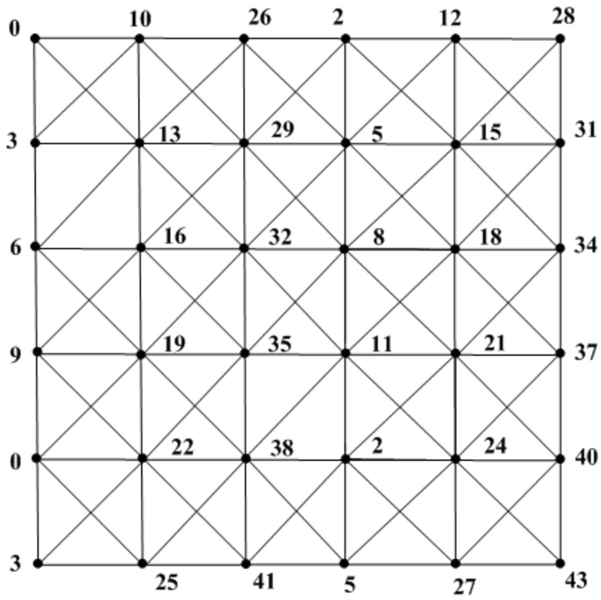
$\urcorner(i, j+1)=\lfloor\frac{n}{2}\rfloor\left(\lceil\frac{n}{2}\rceil-1\right)+\lfloor\frac{n}{2}\rfloor((i-1)+(n-1)(j-1))+j,\ i=1,2, \ldots, n,\ j=1,2, \ldots,\lfloor\frac{n}{2}\rfloor$,

$\urcorner\left(i, \frac{n}{2}+j+1\right)=\lfloor\frac{n}{2}\rfloor\left(\lceil\frac{n}{2}\rceil-1\right)+\lfloor\frac{n}{2}\rfloor((i-1)+(n-1)(j-1)+1)+j,\ i=1,2, \ldots, n, j=1,2, \ldots,\lfloor\frac{n}{2}\rfloor-1$.

See the Figure $4(b)$. The rest of the proof is omitted.



(a)



(b)

Figure 4. A radial radio labeling of KG(n,n) for (a) $n = 6$ and (b) $n = 7$ which illustrates the mapping

4. CONCLUSION

The upper bounds for the radial radio number of chess board graphs $CB(m, n)$ for $m = 2, n$ and the King's graph $KG(m, n)$ for $m = n\,(n > 2)$ has been investigated in this research work. For $m \neq n\,(n > 2)$, $CB(m, n)$ and $KG(m, n)$ is still an open problem. Future this research can be extended to identify higher dimensional networks and study the same radial radio number problem due to its application to telecommunication networks.

REFERENCES

[1] G. Chartrand, D. Erwin and P. Zhang, "A Graph Labeling Problem Suggested by FM Channel Restrictions," *Bull. Inst. Combin. Appl.,* vol. 43, pp. 43.57, 2005.

[2] G. J. Chang and D. Kue, "The L (2,1)-Labeling of Graphs," *SIAM J. Discrete Math.,* vol. 9, pp. 309-316, 1996.

[3] T. Hasunuma, T. Ishii, H. Ono, and Y. Uno, "A Linear Time Algorithm for $L(2,1)$-Labeling of Trees," *Lecture Notes in Computer Science,* vol. 5757, pp. 35-46, 2009, doi: 10.1007/978-3-642-04128-0_4

[4] K. Yenoke, K. M. B. Smitha and T. Maryson, "$L(2,1)$-Labeling of Certain Tree Related Graphs," *Juni Khyat.*, vol. 10, no. 8, pp. 207-212, 2020.

[5] K. Yenoke, D. F. Xavier and T. Maryson, "$L(2,1)$-Labelling of Oxide and Silicate Networks," *Journal of Computer and Mathematical Sciences,* vol. 11, no. 6, pp. 27-33, 2020.

[6] U. M. Prajapati and N. B. Patel, "$L(2,1)$-Labelling of Line Graph of Some Graphs," *Journal of Applied Science and Computations,* vol. 6, no. 5, pp. 304-308, 2019, doi: 16.10089.JASC.2019.V6I5.453459.1500101148.

[7] K. M. B. Smitha and K. Thirusangu, "Distance Two Labeling of Cycle Related Graphs," *International Journal of Pure and Applied Mathematical Sciences,* vol. 9, no. 2, pp. 299-312, 2016.

[8] T. Calamoneri and R. Petreschi, "$L(2,1)$-Labeling of Unigraphs," *Discrete Applied Mathematics* vol. 159, pp. 1196-1206, 2011, doi: 10.1016/j.dam.2011.04.015

[9] D. F. Xavier, K. Yenoke, and S. Preethi, "$L(2,1)$ and $L(3,2,1)$ Labeling of Certain Star and Wheel Related Graphs," *International Journal of Innovative Research in Science, Engineering and Technology,* vol. 9, no. 6, pp. 4712-4716, 2020, doi: 10.15680/IJIRSET.2020.0906004.

[10] B. M. Kim, W. Hwang and B. C. Song, "$L(3,2,1)$-Labeling for the Product of a Complete Graph and a Cycle," *Taiwanese Journal of Mathematics,* vol. 19, no. 3, pp. 849-859, 2016, doi: 10.11650/tjm.19.2015.4632.

[11] S. K. Amanathulla and M. Pal, "$L(3,2,1)$ and $L(4,3,2,1)$-Labeling Problems on Interval Graph," *AKCE International Journal of Graph and Combinatorics,* vol.14, pp. 205-215, 2017, doi: 10.1016/j.akcej.2017.03.002.

[12] G. Chartrand, D. Erwin, and P. Zhang, "Radio Labeling of Graphs," *Bull. Inst. Combin. Appl.,* vol. 33, pp. 77-85, 2001.

[13] B. Rajan, I. Rajasingh, K. Yenoke, and P. Manuel, "Radio Number of Graphs with Small Diameter," *International Journal of Mathematics and Computer Science,* vol. 2, no. 3, pp. 209-220, 2007.

[14] B. Rajan and K. Yenoke, "On the Radio Number of the Hexagonal Mesh," *Journal of Combinatorial Mathematics and Combinatorial Computing,* vol. 79, pp. 235-244, 2011.

[15] S. K. Vaidya and D. D. Bantva, "Radio Number for Total Graph of Paths," *International Scholarly Research Notices,* vol. 2013, pp. 1-6, 2013, doi: 10.1155/2013/326038.

[16] K. Yenoke and M. K. A. Kaabar, "The Bounds for the Distance Two Labelling and Radio Labelling of Nnostar Tree Dendrimer, *Telkomnika Telecommunications, Computing, Electronics and Control,* Preprint. (X)

[17] G. Chartrand, D. Erwin, and P. Zhang, "Radio Antipodial Colorings of Graphs," *Mathematica Bohemica,* vol. 127, no. 1, pp. 57-69, 2002, doi: 10.21136/MB.2002.133978.

[18] G. Chartrand, D. Erwin, and P. Zhang, "Radio Antipodal Coloring of Cycles," *Cong. Numer.,* vol. 144, pp. 129-141, 2000. (X)

[19] A. William and C. R. Kenneth, "Radio Antipodal Number of Certain Graphs," *ICIEIS 2011: Informatics Engineering and Information Science,* vol. 253, pp. 385-389, 2011, doi: 10.1007/978-3-642-25462-8_34.

[20] L. Saha and P. Panigrahi, "Antipodal Number of Some Powers of Cycles," *Discrete Mathematics,* vol. 312, no. 9, pp. 1550-1557, 2012, doi: 10.1016/j.disc.2011.10.032.

[21] S. Avadayappan, M. Bhuvaneshwari, and S. Vimalajenifer, "A Note on Radial Radio Number of a Graph," *International Journal of Applied and Advanced Scientific Research,* Special Issue, February, pp. 62-68, 2017.

[22] S. Avadayappan, M. Bhubaneshwar, and S. Vimalajenifer, "The Radial Radio Number and the Clique Number of a Graph," *International Journal of Engineering and Advanced Technology (IJEAT),* vol. 9, no. 1S4, pp. 1002-1006, 2019, doi: 10.35940/ijeat.A1215.1291S419.

[23] K. Yenoke, "Radial Radio Number of Cyclic and Wheel Split Graphs," *International Journal of Advance Research, Ideas and Innovations in Technology*, vol. 6, no. 4, pp.222-227, 2020.

[24] T. A. Jose and M. Giridharan, "Radial Radio Mean Labeling of Mongolian Tent and Diamond Graphs," *International Journal for Research in Applied Science and Engineering Technology,* vol. 8, no. 7 pp. 2078-2083, 2020, doi: 10.22214/ijraset.2020.30759.

[25] P. M. Pardalos, D.-Z. Du, and R. L. Graham, "Handbook of Combinatorial Optimization," *Springer Science & Business Media,* vol. 2, pp. 1-394, 2006, doi: 10.1007/978-1-4419-7997-1.

[26] G. Chartrand, L. Nebesky, and P. Zhang, "Radio $k-$colorings of paths," *Discussiones Mathematicae Graph Theory,* vol. 24, no. 1, pp. 5-21, 2004, doi: 10.7151/dmgt.1209.

[27] D. Sakai, "Labeling Chordal Graphs with a Condition at Distance Two," *SIAM Journal on Discrete Mathematics,* vol. 7, no. 1, pp. 133-140, 1994, doi: 10.1137/S0895480191223178.

[28] R. Ponraj and S. S. Narayanan, "On Radio Mean Number of Some Graphs," *International J. Math. Combin.,* vol. 3, pp. 41-48, 2014.

BIOGRAPHIES OF AUTHORS

Dr. Kulandaivel Maruthamuthu Paramasivam received the Doctor of Philosophy (Ph.D) in Mathematics from Anna University, Chennai, Tamilnadu, India. Obtained his Bachelor (B.Sc) and Master of Philosophy (M.Phil) from St. Joseph's College (Autonomous), Tiruchirappalli, Tamilnadu, India. Received his Master's degree (M.Sc) from Loyola College (Autonomous), Chennai, Tamilnadu, India. He Actively involved in research for the past fifteen years. His area of research interest is Graph theory, Ad Hoc Networks and Internet of Things. Currently working as a Professor of Mathematics Section, Department of Information Technology, University of Technology and Applied Sciences – Al Mussanah, Sultanate of Oman. During these 20 years of teaching, he has successfully taken up a number of teaching and administrative assignments. Also serving as an Executive Editor of International Journal of Applied Graph Theory https://www.ijagt.com/editorial-team/. He can be contacted at email: mpkoman@gmail.com.

Dr. Kins Yenoke received his UG and PG degrees from Manonmaniam Sundaranar University, Tirunelveli, Tamilnadu, India. He received his M.Phil. and Ph.D. in Mathematics from Loyola College, Chennai, India. He is currently working as an Assistant Professor in the PG and Research department of Loyola College, Chennai, India. His areas of research are channel assignment problems, graph labelling problems in graph theory. Along with 13 years of teaching experience, he has played a key role in organizing many international conferences, seminars, workshops etc. His passion involves enlightening the young minds to excel in the current research areas and motivating the staff members to write research articles through faculty development programmes. He also received various prestigious Awards in the Area of Research and Student development. He can be contacted at email: jecinthokins@rediffmail.com.

Dr. Baby Smitha Kanaka Muralidharan received the Doctor of Philosophy (Ph.D) in Mathematics from Madras University, Chennai, Tamilnadu, India. She holds an active teaching experience of fifteen years and research experience of seven years in the field of Graph Theory. She is currently working as Assistant Professor in the Department of Mathematics, Devaswom Board College, Thalayolaparambu, Kottayam, Kerala, India. Her major research contribution concentrates on applications of radio mean labeling in communication and wireless networks. She is a life member of Ramanujan Mathematical Society since 2013. She can be contacted at email: smithavenu82@gmail.com.