

The bounds for the distance two labelling and radio labelling of nanostar tree dendrimer

Kins Yenoke¹, Mohammed K. A. Kaabar²

¹Department of Mathematics, Loyola College, Chennai, India

²Institute of Mathematical Sciences, Faculty of Science, University of Malaya, Kuala Lumpur, Malaysia

Article Info

Article history:

Received Aug 29, 2020

Revised Dec 17, 2021

Accepted Dec 26, 2021

Keywords:

Distance-two-labelling

Labelling

Radio labelling

Radio number

ABSTRACT

The distance two labelling and radio labelling problems are applicable to find the optimal frequency assignments on AM and FM radio stations. The *distance two labelling*, known as $L(2,1)$ -labelling of a graph A , can be defined as a function, k , from the vertex set $V(A)$ to the set of all non-negative integers such that $d(c, s)$ represents the distance between the vertices c and s in A where the absolute values of the difference between $k(c)$ and $k(s)$ are greater than or equal to both 2 and 1 if $d(c, s)=1$ and $d(c, s) = 2$, respectively. The $L(2,1)$ -labelling number of A , denoted by $\lambda_{2,1}(A)$, can be defined as the smallest number j such that there is an $L(2,1)$ -labelling with maximum label j . A *radio labelling* of a connected graph A is an injection k from the vertices of A to N such that $(c, s) + |k(c) - k(s)| \geq 1 + d \quad \forall c, s \in V(A)$, where d represents the diameter of graph A . The *radio numbers* of k and A are represented by $rn(k)$ and $rn(A)$ which are the maximum number assigned to any vertex of A and the minimum value of $rn(k)$ taken over all labellings k of A , respectively. Our main goal is to obtain the bounds for the distance two labelling and radio labelling of nanostar tree dendrimers.

This is an open access article under the [CC BY-SA](https://creativecommons.org/licenses/by-sa/4.0/) license.



Corresponding Author:

Kins Yenoke

Department of Mathematics, Loyola College

Chennai-034, Tamil Nadu, India

Email: jecinthokins@rediffmail.com

1. INTRODUCTION

In the field of communication engineering, the radio frequencies are commonly used in communication devices such as radio transmitters, computers, televisions, and mobile phones due to the fact that the frequency and energy of radio waves are very low. Researchers and engineers are working on optimizing the usage of the allotted bandwidth for a specified communication system due to the high cost of spectrum. In 1992, Griggs and Yeh [1] optimized the number of channels for the amplitude modulation (AM) radio stations in the stipulated bandwidth with the help of a graph labelling technique, known as distance two labelling. Motivated by the distance two labelling concept, Chartrand *et al.* [2] introduced in the early 21st century the radio labelling concept for the frequency modulation (FM) radio stations. This type of channel allocation concerns with the maximum number of channels in a particular geographical area such that all the stations can receive the distinct frequencies. Since the distance between transmitters and their difference in frequency has played a vital role in assigning the maximum number of channels, the distance two labelling and radio labelling can be defined as follows: The distance two labelling, denoted by $L(2,1)$ -labelling of a graph A , is a function, k , from the vertex set $V(A)$ to the set of all non-negative integers such that $d(c, s)$ represents the distance between the vertices c and s in A ; therefore, we have $|k(c) - k(s)| \geq 2$ and

$|k(c) - k(s)| \geq 1$ if $d(c, s) = 1$ and $d(c, s) = 2$, respectively. The $L(2,1)$ -labelling number of A , denoted by $\lambda_{2,1}(A)$, can be defined as the smallest number j such that there is a $L(2,1)$ -labelling with maximum label j . On one hand, Fotakis *et al.* [3] proved the NP-hardness of the radio coloring problem for graphs with diameter 2. On the other hand, Fiala *et al.* [4] investigated the NP-completeness for series-parallel graphs. Havet *et al.* [5] established the optimal exact algorithm for $L(2,1)$ -labelling as $O(3.8730^n)$ via dynamic programming. However, Szaniawski *et al.* [6] improved this bound by $O^*(3.5616^n)$. By using the algorithm proposed by Chang and Kuo [7], the upper bound $\lambda_{2,1}(A) \leq \Delta^2 + \Delta - 2$ was determined by Goncalves [8]. Bodlaender *et al.* [9] showed that, for a given permutation graph A , $\lambda_{2,1}(A) \leq 5\Delta - 2$ is obtained. In addition, Sakai [10] obtained the distance two labelling of chordal graphs. Smitha and Thirusangu [11] proved the results for the quadrilateral snake Q_n as 8 and for the alternate quadrilateral snake graph Q_n as 5 for $n \geq 2$. Kujur *et al.* [12] proved that $\lambda_{2,1}(B_{m,n}) \leq 13$, where the bloom graph is $B_{m,n}(n, m > 2)$. Furthermore, Yenoke *et al.* [13] found the bounds for silicate and oxide networks as $\lambda_{2,1}(OX(n)) \leq 8$ and $\lambda_{2,1}(SL(n)) \leq 10$, respectively.

For a connected graph A , radio labelling is an injection, k , from the vertices of A to N such that d represents the diameter of a graph A , the result, $d(c, s) + |k(c) - k(s)| \geq 1 + d \forall c, s \in V(A)$, is obtained. The radio numbers of k and A are represented by $rn(k)$ and $rn(A)$ which are the maximum number assigned to any vertex of A and the minimum value of $rn(k)$ taken over all labellings k of A , respectively. The following Figure 1 depicts the definition of radio number.

For the last two decades, several authors studied the radio labelling problem for general graphs and certain interconnection networks. The radio number of the total path of graphs were determined by Vaidya and Bantva [14]. Cada *et al.* [15] obtained the radio number of distance graphs. The same problem for trees was studied by Liu [16]. Kim *et al.* [17] presented the product of graphs namely P_l ($l \geq 4$) and K_l ($l \geq 2$). Bharati and Yenoke [18] determined both upper and lower bounds for the hexagonal mesh as $n(3n^2 - 4n - 1) + 3$ and $3n^2 - 3n + 2 + \sum_{i=0}^{n-2} i(n - i - 1)$, respectively. Bantva [19] slightly improved the lower bound that was established in [20]. In addition, Yenoke *et al.* [21] proved that $r_n(EN(n, n)) \leq (n - 2)(4n^2 - 9n + 8) + 2(n - 1)^2 + (n + 1)$, where $EN(n, n)$ is the enhanced mesh, $n \geq 4$. This paper is divided as follows: In section 2, we discuss the methodology of our research work. In section 3, our main results are obtained by studying the bounds for the $L(2,1)$ -labelling number and radio number of the general tree dendrimer $T_{n,p}$. Our research work is concluded in section 4.

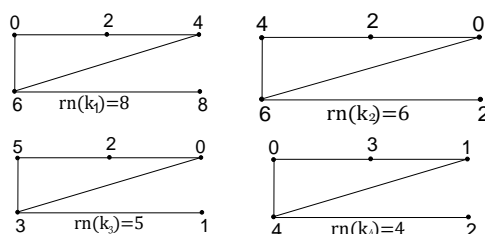


Figure 1. Different radio labellings and radio number of a graph A , $rn(A) = \min\{8, 6, 5, 4\} = 4$

2. RESEARCH METHOD

The author studied in [18], [21] the same problem for the networks that contains the number of vertices in the n^{th} dimension as $3n^2 - 3n + 1$ and n^2 , respectively. In addition, the authors in [14], [15], [17] studied the same problem for the graphs with n vertices. Since the vertices have been increased in terms of the high order for a network, especially of order n^n ; therefore, finding a good solution is very complicated. The authors are trying to find a solution for such networks. However, a good bound is obtained in this paper for the tree dendrimer chemical network which grows in the order of generation. Further, most of the networks were studied separately for the $L(2,1)$ labelling or radio labelling. Due to the exponential growth of communication technology, we are today in need to estimate the lower and upper bounds for the graphs that are growing in higher order to compete with the consumers' demand. By taking this into our account and according to the best of our knowledge, for the first time ever, we have estimated in this research work the bounds for both $L(2,1)$ -labelling and radio labelling numbers for an n^n (number of vertices) expanding chemical structure, known as nanaostar tree dendrimer. This research study provides a detailed analysis of the growth of such graph in terms of diameter and vertices for $n, p > 2$, and its bounds have been obtained separately for $L(2,1)$ -labelling number and radio labelling number. Therefore, all obtained results in this study are novel and worthy.

2.1. Nanostar tree dendrimer

Nanostar is a star-looking type of nanoparticle that contains a spherical core with many branches. Dendrimers have very complex chemical structures and hyper-branched macromolecules with a star-shaped architecture. In addition, dendrimers are classified by a generation which represents the repeated branching cycles number that are performed during its synthesis. The structure of these materials has a huge impact on the physical and chemical properties of dendrimers due to the uniqueness of dendrimers' behavior which makes them very suitable for various biomedical and industrial applications [22]-[24]. Yang and Xia [24] defined a tree dendrimer graph, denoted by $T_{n,p}$, as follows: The center vertex of the graph $T_{n,p}$ is represented by v_1^0 which is a p -regular graph except the pendant vertices. In addition, the distance from the center vertex v_1^0 to every pendant vertex is exactly n . Moreover, n signifies here the n^{th} generation of the tree dendrimer. The diameter and radius of a tree dendrimer graph are $2n$ and n , respectively.

In this research work, we have named the n generation vertices of the tree dendrimer $T_{n,p}$ as follows: First, we name the p vertices in the first generation which are adjacent to the center vertex v_1^0 as $v_1^1, v_2^1 \dots v_p^1$ in the clockwise sense. Next, we name the $p(p-1)$ vertices in the second generation as $v_1^2, v_2^2 \dots v_{p(p-1)}^2$ in the same order as we did the previous step. Similarly, we name the vertices of $3^{\text{rd}}, 4^{\text{th}} \dots (n-1)^{\text{th}}$ generations. Finally, the $p(p-1)^{n-1}$ vertices in the n^{th} generation are named as $v_1^n, v_2^n \dots v_{p(p-1)^{n-1}}^n$ as shown in Figure 2.

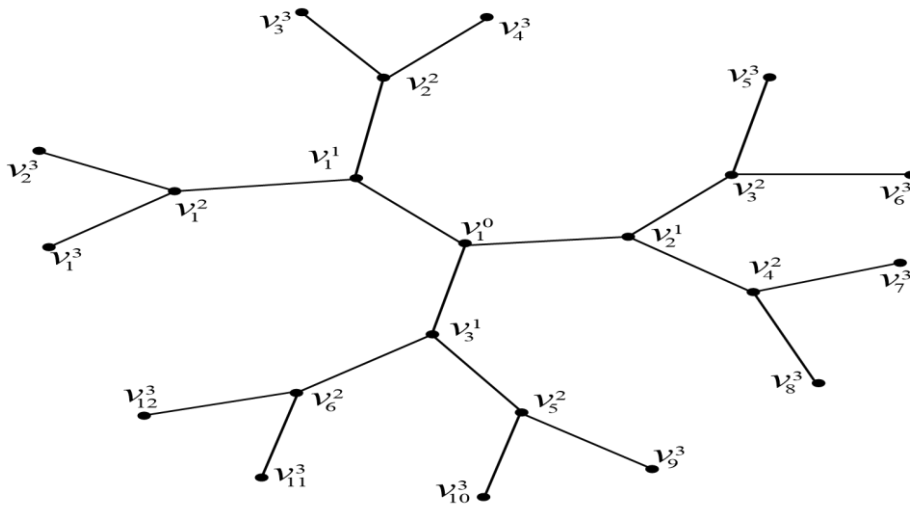


Figure 2. Vertices' Naming in $T_{3,3}$

3. RESULTS AND ANALYSIS

The bounds for the $L(2,1)$ -labelling number and radio number of the general tree dendrimer $T_{n,p}$ are determined in this section.

Proposition 3.1: For any connected simple graph A of diameter 2, $\lambda_{2,1}(A) + 1 = rn(A)$.

Proof: The proof is directly derived from the definitions of $L(2,1)$ -labelling and radio labelling.

a) Theorem 3.1: For $p > 2$, $\lambda_{2,1}(T_{1,p}) + 1 = rn(T_{1,p}) = p + 2$.

Proof: It is known that the tree dendrimer $T_{1,p}$ is an ordinary star graph S_{p+1} . Rajan *et al.* [25] showed that the radio number of a star graph is $rn(S_{p+1}) = p + 2, p > 2$. Since the diameter of $T_{1,p}$ is 2, from Proposition 3.1, we obtain $\lambda_{2,1}(T_{1,p}) + 1 = rn(T_{1,p}) = p + 2, p > 2$.

b) Theorem 3.2: For $n = 2$ and $p > 2$, the $L(2,1)$ labelling number of $T_{n,p}$ satisfies, $\lambda_{2,1}(T_{2,p}) \leq 2p$.

Proof: By defining a mapping $k: V(T_{2,p}) \rightarrow N$, we have the following:

$$k(v_1^0) = 1$$

$$k(v_{(j-1)(p-1)+i}^2) = i + 1, i = 1, 2 \dots p - 1, j = 1, 2 \dots p$$

$$k(v_i^1) = p + 1 + i, i = 1, 2 \dots p.$$

Next, we claim that k is a valid radio 2-chromatic labelling.

Let c and s be any two vertices in $T_{2,p}$.

1) Case 1: Assume c and s are any two second generation vertices.

If $c = v_{(z-1)(p-1)+j}^2$ and $s = v_{(z-1)(p-1)+l}^2$, $1 \leq j \neq l \leq p-1$, then we have $d(c, s) = 2$ and

$|k(c) - k(s)| = 1$. Therefore, we have: $d(c, s) + |k(c) - k(s)| \geq 3$.

In addition, if $c = v_{(j-1)(p-1)+i}^2$ and $s = v_{(l-1)(p-1)+i}^2$, $1 \leq j \neq l \leq p$; hence, the distance between them is 4.

Therefore, the condition is trivially satisfied.

2) Case 2: Suppose c and s are the first-generation vertices, then $d(c, s) = 2$ and $k(c) = p + 1 + j, k(s) = p + 1 + l$, $1 \leq j \neq l \leq p$. Therefore, we obtain $d(c, s) + |k(c) - k(s)| \geq 2 + |j - l| \geq 3$.

3) Case 3: Suppose c is a second-generation vertex and s is a first-generation vertex, then we have: $d(c, s) \geq 1$ and $k(c) = j + 1, k(s) = p + 1 + l$, $1 \leq j \leq p - 1$, $1 \leq l \leq p$. Therefore, $d(c, s) + |k(c) - k(s)| \geq 1 + |p| \geq 3$.

4) Case 4: If c is the center vertex and $v = v_{(k-1)(p-1)+i}^2$, then $d(c, s) = 2$ and $|k(c) - k(s)| \geq 1$. Therefore, we obtain: $d(c, s) + |k(c) - k(s)| \geq 3$.

5) Case 5: If $c = v_1^0$ and $v = v_i^1$, then $k(c) = 1, k(s) \geq p + 2$. Therefore, $d(c, s) + |k(c) - k(s)| \geq p + 3 > 3$.

Hence, the $L(2,1)$ labelling condition is satisfied, and the vertex v_p^1 attains the maximum value $2p + 1$. Since we start labelling from 1, we get $\lambda_{2,1}(T_{2,p}) + 1 \leq 2p + 1$, which implies that $\lambda_{2,1}(T_{2,p}) \leq 2p$.

c) Theorem 3. 3: Let A be a tree dendrimer $T_{n,p}$, where $p, n > 2$, then the $L(2,1)$ labelling number of A satisfies $\lambda_{2,1}(A) \leq 3p + 1$.

Proof: Define a mapping $k: V(T_{n,p}) \rightarrow N$ as follows:

$$k(v_1^0) = 1.$$

$$k(v_i^1) = i + 2, i = 1, 2 \dots p.$$

$$k(v_{(z-1)(p-1)+i}^{3j-1}) = p + 3 + i, i = 1, 2 \dots p - 1, z = 1, 2 \dots (p - 1)^{3(j-1)}p, j = 1, 2 \dots \left\lfloor \frac{n}{3} \right\rfloor.$$

$$k(v_{(z-1)(p-1)+i}^{3j}) = 2p + 3 + i, i = 1, 2 \dots p - 1, z = 1, 2 \dots (p - 1)^{3j-2}p, j = 1, 2 \dots \left\lfloor \frac{n}{3} \right\rfloor.$$

$$k(v_{(z-1)(p-1)+i}^{3j+1}) = i + 2, i = 1, 2 \dots p - 1, z = 1, 2 \dots (p - 1)^{3j}p, j = 1, 2 \dots \left\lfloor \frac{n}{3} \right\rfloor - 1 \text{ as shown in}$$

Figure 3.

Next, we verify the radio 2-chromatic labelling condition $d(c, s) + |k(c) - k(s)| \geq 3 \forall c, s \in V(T_{n,p})$.

By letting $c, s \in V(T_{n,p})$, we have the following:

- 1) Case 1: Suppose c and s are any two vertices in $(3j - 1)^{th}$ generation, where $j = 1, 2 \dots \left\lfloor \frac{n}{3} \right\rfloor$.
 - Case 1.1: If $c = v_{(p-1)(z-1)+i}^{3j-1}$ and $s = v_{(p-1)(z-1)+i}^{3l-1}$, $1 \leq j \neq l \leq \left\lfloor \frac{n}{3} \right\rfloor$, then we have: $d(c, s) \geq 3$ and $|k(c) - k(s)| \geq 0$. Therefore, we obtain: $d(c, s) + |k(c) - k(s)| \geq 3$.
 - Case 1.2: If $c = v_{(p-1)(w-1)+i}^{3j-1}$ and $s = v_{(p-1)(t-1)+i}^{3j-1}$, $1 \leq w \neq t \leq (p - 1)^{3(j-1)}p$, then we have: $d(c, s) \geq 4$ and $|k(c) - k(s)| \geq 0$. Therefore, we obtain $d(c, s) + |k(c) - k(s)| > 3$.
 - Case 1.3: If $c = v_{(p-1)(z-1)+w}^{3j-1}$ and $s = v_{(p-1)(z-1)+t}^{3j-1}$, $1 \leq w \neq t \leq p - 1$, then we have:
 - $d(c, s) = 2$ and $|k(c) - k(s)| = |(p + 3 + w) - (p + 3 + t)| = |w - t| \geq 1$, since $w \neq t$.
 - Hence, we have $d(c, s) + |k(c) - k(s)| \geq 2 + 1 = 3$.
- 2) Case 2: Let us assume that c and s lie in the $(3j)^{th}$ generation, where j varies from 1 to $\left\lfloor \frac{n}{3} \right\rfloor$.
 - Case 2.1: If c and s are of the form $v_{(p-1)(z-1)+i}^{3j}$ and $v_{(p-1)(z-1)+i}^{3l}$, $1 \leq j \neq l \leq \left\lfloor \frac{n}{3} \right\rfloor$, then the distance between them is at least 3, which directly verifies the radio 2-chromatic labelling condition.
 - Case 2.2: If $c = v_{(p-1)(w-1)+i}^{3j}$ and $s = v_{(p-1)(t-1)+i}^{3k}$, $1 \leq w \neq t \leq (p - 1)^{3j-2}p$, then $d(c, s) \geq 4$. Therefore, $d(c, s) + |k(c) - k(s)| > 3$.
 - Case 2.3: If $c = v_{(p-1)(z-1)+w}^{3j}$ and $s = v_{(p-1)(z-1)+t}^{3j}$, $1 \leq w \neq t \leq p - 1$, then $d(c, s) = 2$ and $|k(c) - k(s)| = |(2p + 3 + w) - (2p + 3 + t)|$. Hence, $d(c, s) + |k(c) - k(s)| \geq 2 + |w - t| \geq 3$, since $w \neq t$.

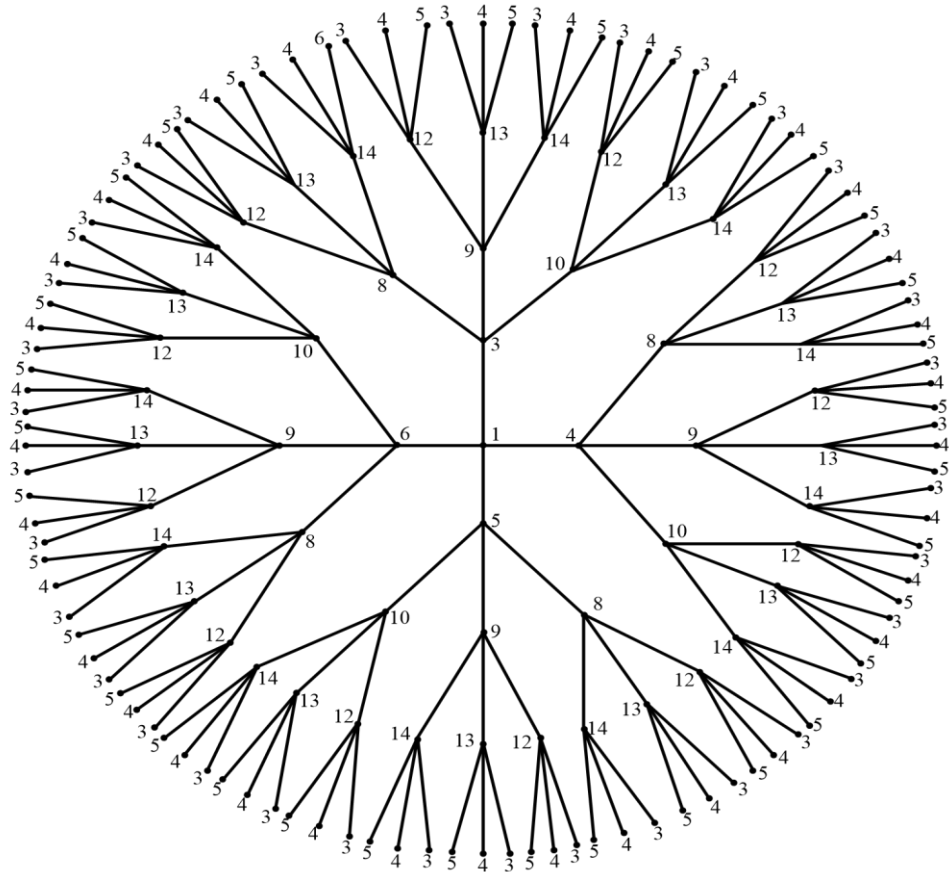


Figure 3. Radio 2-chromatic labelling of $T_{n,p}$ with $n = p = 4$ which attains the upper bound

- 3) Case 3: Assume that c and s are $(3j + 1)^{th}$ generation vertices, where $j = 1, 2 \dots \lfloor \frac{n}{3} \rfloor$.
 - Case 3.1: If $c = v_{(p-1)(z-1)+i}^{3j+1}$ and $s = v_{(p-1)(z-1)+i}^{3l+1}$, $1 \leq j \neq l \leq \lfloor \frac{n}{3} \rfloor - 1$, then $|k(c) - k(s)| \geq 0$ and the distance between them is greater than 2. Therefore, $d(c, s) + |k(c) - k(s)| \geq 3$.
 - Case 3.2: If $c = v_{(p-1)(w-1)+i}^{3j+1}$ and $s = v_{(p-1)(t-1)+i}^{3j+1}$, $1 \leq w \neq t \leq (p - 1)^{3j}p$, then we have: $d(c, s) \geq 4$, which trivially verifies the labelling condition.
 - Case 3.3: If $c = v_{(p-1)(z-1)+w}^{3j-1}$ and $s = v_{(p-1)(z-1)+t}^{3j-1}$, then we have $k(c) = w + 2$ and $k(s) = t + 2$, $1 \leq w \neq t \leq p - 1$. In addition, the distance between them is exactly 2. Hence, we obtain the following: $d(c, s) + |k(c) - k(s)| \geq 3$.
- 4) Case 4: Suppose c and s are first generation vertices then, $k(c) = w + 2$ and $k(s) = t + 2$, $1 \leq w \neq t \leq p$. In addition, in this case $d(c, s) = 2$. Therefore, $d(c, s) + |k(c) - k(s)| \geq 2 + |w - t| \geq 3$ since $w \neq t$.
- 5) Case 5: Suppose $c = v_i^1$ and $c = v_{(p-1)(z-1)+i}^{3j-1}$, then $k(c) = i + 2$, $1 \leq i \leq p$ and $k(s) = p + 3 + i$, $1 \leq i \leq p - 1$. In addition, $d(c, s) \geq 1$. Hence, $d(c, s) + |k(c) - k(s)| \geq 1 + |(p + 3 + 1) - (p + 2)| = 3$.
- 6) Case 6: Suppose $c = v_i^1$, $1 \leq i \leq p$ and $c = v_{(p-1)(z-1)+i}^{3j}$, $1 \leq i \leq p - 1$, $1 \leq z \leq (p - 1)^{3(j-1)}p$, $1 \leq j \leq \lfloor \frac{n}{3} \rfloor$, then $k(c) = 2 + i$, and $k(s) = 2p + 3 + i$. Also, the distance between them is at least 2. Therefore, $d(c, s) + |k(c) - k(s)| \geq 2 + |(2p + 3 + 1) - (p + 2)| = p + 4 > 3$.
- 7) Case 7: Suppose $c = v_i^1$, $1 \leq i \leq p$ and $c = v_{(p-1)(z-1)+i}^{3j+1}$, $1 \leq i \leq p - 1$, $1 \leq z \leq (p - 1)^{3j}p$, $1 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1$, then $|k(c) - k(s)| \geq 0$. However, the distance between them is at least 3. Hence, we obtain: $d(c, s) + |k(c) - k(s)| \geq 3$.

- 8) Case 8: Suppose $c = v_{(p-1)(z-1)+i}^{3j-1}$ and $c = v_{(p-1)(z-1)+i}^{3j}$, $1 \leq i \leq p-1$, then we have $k(c) = p+3+i$, $k(s) = 2p+3+i$ and $d(c, s) \geq 1$. Therefore, we obtain: $d(c, s) + |k(c) - k(s)| \geq 1 + |(2p+3+1) - (2p+2)| = 3$.
- 9) Case 9: Suppose $c = v_{(p-1)(z-1)+i}^{3j-1}$ and $c = v_{(p-1)(z-1)+i}^{3j+1}$, $1 \leq i \leq p-1$, then $k(c) = p+3+i$, $k(s) = i+2$ and $d(c, s) \geq 2$. Therefore, we have: $d(c, s) + |k(c) - k(s)| \geq 2 + |(p+4) - (p+1)| > 3$.
- 10) Case 10: If $c = v_{(p-1)(z-1)+i}^{3j}$ and $c = v_{(p-1)(z-1)+i}^{3j+1}$, $1 \leq i \leq p-1$, then we get: $k(c) = 2p+3+i$, $k(s) = i+2$ and $d(c, s) \geq 2$. Therefore, $d(c, s) + |k(c) - k(s)| \geq 2 + |(2p+4) - (p+1)| = p+5 > 3$.
- 11) Case 11: If c is the center vertex v_1^0 , and s is any other vertex in the graph. Then, $d(c, s) \geq 1$ and $|k(c) - k(s)| \geq 2$. Therefore, we obtain: $d(c, s) + |k(c) - k(s)| \geq 3$.

Hence, $d(c, s) + |k(c) - k(s)| \geq 3$ for every pair of vertices c and s in $V(T_{n,p})$.

In addition, the vertices $v_{z(p-1)}^{3j}$, $z = 1, 2 \dots (p-1)^{3j-2}$, $j = 1, 2 \dots \lfloor \frac{n}{3} \rfloor$ attains the maximum value $3p+2$, which implies that $\lambda_{2,1}(A) + 1 \leq 3p+2$.

Thus, $\lambda_{2,1}(A) \leq 3p+1$.

Remark 1: The branches of the tree dendrimer which are connected to the center vertex v_1^0 by a single edge are called the main branches of the tree dendrimer graph. We denote the p main branches in $T_{n,p}$ as B^j , $j = 1, 2 \dots p$.

Lemma 3.1: Let $T_{n,p}$ be a tree dendrimer graph of n generations with each vertex of degree p expects the pendant vertices, then the number of vertices in each main branch B^j ($1 \leq j \leq p$) is $\frac{(p-1)^n - 1}{p-2}$.

Proof: From the construction of the tree dendrimer, the first generation contains $p-1$ vertices. Since the root vertex of a main branch is a vertex of first generation, there is only a single vertex in the first generation. Therefore, the number of vertices in a second generation is $p-1$. In general, the number of vertices in the n^{th} generation is $(p-1)^{n-1}$. Hence, the total number of vertices in a main branch is calculated as follows:

$$1 + (p-1) + (p-1)^2 + \dots + (p-1)^{n-1} = \frac{(p-1)^n - 1}{p-1-1} = \frac{(p-1)^n - 1}{p-2}.$$

d) Theorem 3.4: Let $T_{n,p}$ ($n, p > 2$) be a tree dendrimer graph of diameter $2n$. Then, an upper bound for the radio number of $T_{n,p}$ is given by $rn(T_{n,p}) \leq n + (2n-1)p + 1 + \sum_{l=1}^{n-1} (2l-1)p((p-1)^{n-l-1})(p-2) + (2l-1)p(p)^{n-l-1} - 1$, whenever $p \geq 2n-3$.

Proof: Define a mapping h from the vertex set of $T_{n,p}$ to the natural numbers as follows:

First, we label the center vertex v_1^0 as $h(v_1^0) = 1$. Since the pendant vertices are at a distance n from the center vertex, we label the n^{th} generation pendant vertices in B^j , $j = 1, 2 \dots p$ as $k\left(v_{(p-1)^{n-1}(j-1)+z+(p-1)(i-1)}^n\right) = (p(p-1)^{n-2})(z-1) + pi - 1$, $i = 1, 2 \dots (p-1)^{n-2}$, $z = 1, 2 \dots p-1$.

Next, we label the $(n-1)^{\text{th}}$ generation vertices in B^j , $j = 1, 2 \dots p$ as

$$k\left(v_{(p-1)^{n-2}(j-1)+z+(p-1)(i-1)}^{n-1}\right) = (p(p-1)^{n-2})(p-2) + p((p-1)^{n-2}) - 1 + (3p((p-1)^{n-3}))(z-1) + 3pi - 1, \quad i = 1, 2 \dots (p-1)^{n-3}, \quad z = 1, 2 \dots p-1.$$

In general, $(n-l)^{\text{th}}$ ($1 \leq l \leq n-1$) generation vertices in B^j , $j = 1, 2 \dots p$ are labelled as

$$k\left(v_{(p-1)^{n-l}(j-1)+z+(p-1)(i-1)}^{n-l}\right) = k\left(v_{(p-1)^{n-l+1}+p-1+(p-1)((p-1)^{n-l-3}-1)}^{n-l+1}\right) + (2l-1)p((p-1)^{n-l-2})(z-1) + (2l-1)pi - 1, \quad i = 1, 2 \dots (p-1)^{n-l-2}, \quad z = 1, 2 \dots p-1.$$

Finally, let us label the first-generation vertices of $T_{n,p}$ as $k(v_i^1) = k(v_{p(p-1)}^2) + i(2n-1) - 1$,

$i = 1, 2 \dots p$ as shown in Figure 4.

Given the diameter of the graph is $2n$, to prove k is a valid radio labelling, we must verify the condition $d(c, s) + |k(c) - k(s)| \geq 2n+1$ for every pair of vertices c and s in $T_{n,p}$.

- 1) Case 1: Assume c and s are any two vertices in the same B^j ($1 \leq j \leq p$), then we have: $d(c, s) = 2i$, $i = 1, 2 \dots n-1$. Therefore, for such cases, $|k(c) - k(s)| \geq 2n-2i+1$ since $p \geq 2n-3$. Hence, we obtain: $d(c, s) + |k(c) - k(s)| \geq |2i - (2n-2i+1)| = 2n+1$.
- 2) Case 2: Assume c is a vertex of B^w and s is a vertex of B^t , $1 \leq w \neq t \leq p$, then the distance between them is exactly $2n$, and $|k(c) - k(s)| \geq n$. Therefore, we have: $d(c, s) + |k(c) - k(s)| \geq 2n+1$.
- 3) Case 3: Assume c is the center vertex and s is any other vertex in $T_{n,p}$.
 - Case 3.1: If c is a pendant vertex, then $d(c, s) = n$ and $k(c) = 1$, $k(s) \geq n+1$. Therefore, in this sub case, $d(c, s) + |k(c) - k(s)| \geq 2n+1$.
 - Case 3.2: If s is not a pendant vertex, then $d(c, s) \geq 1$ and $k(c) = 1$, $k(s) \geq (p-1)^2 + n+1$. Since $p \geq 2n-3$, we obtain: $d(c, s) + |k(c) - k(s)| \geq 2n+1$.

Thus, k is a valid radio labelling and the vertex $k(v_p^1)$ attains the maximum value $n + (2n - 1)p + 1 + \sum_{l=1}^{n-1} (2l - 1)p((p - 1)^{n-l-1})(p - 2) + (2l - 1)p(p)^{n-l-1} - 1$.

Hence, the radio number of $T_{n,p}$ ($n, p > 2$) satisfies $rn(T_{n,p}) \leq n + (2n - 1)p + 1 + \sum_{l=1}^{n-1} (2l - 1)p((p - 1)^{n-l-1})(p - 2) + (2l - 1)p(p)^{n-l-1} - 1$, whenever $p \geq 2n - 3$.

Hence, the theorem is proven.

Next, we determine the lower bound for the radio number of $T_{n,p}$ ($n, p > 2$) by using the following theorem which was proven by Bharati and Yenoke [18]:

Theorem 3.5 (As Theorem 2 in [18]): Let A be a simple connected graph of order m . Let $m_0, m_1 \dots m_j$ be the number of vertices that have eccentricities $e_0, e_1 \dots e_j$, where $diam(A) = d = e_0 > e_1 > \dots > e_j = rad(A)$. Then, we obtain the following:

$$rn(A) \geq \begin{cases} m - 2(d - e_j) + \sum_{i=1}^k 2(d - e_i)m_i, & \text{if } m_j > 1 \\ m - (d - e_j) - (d - e_{j-1}) + \sum_{i=1}^j 2(d - e_i)m_i, & \text{if } m_j = 1 \end{cases}$$

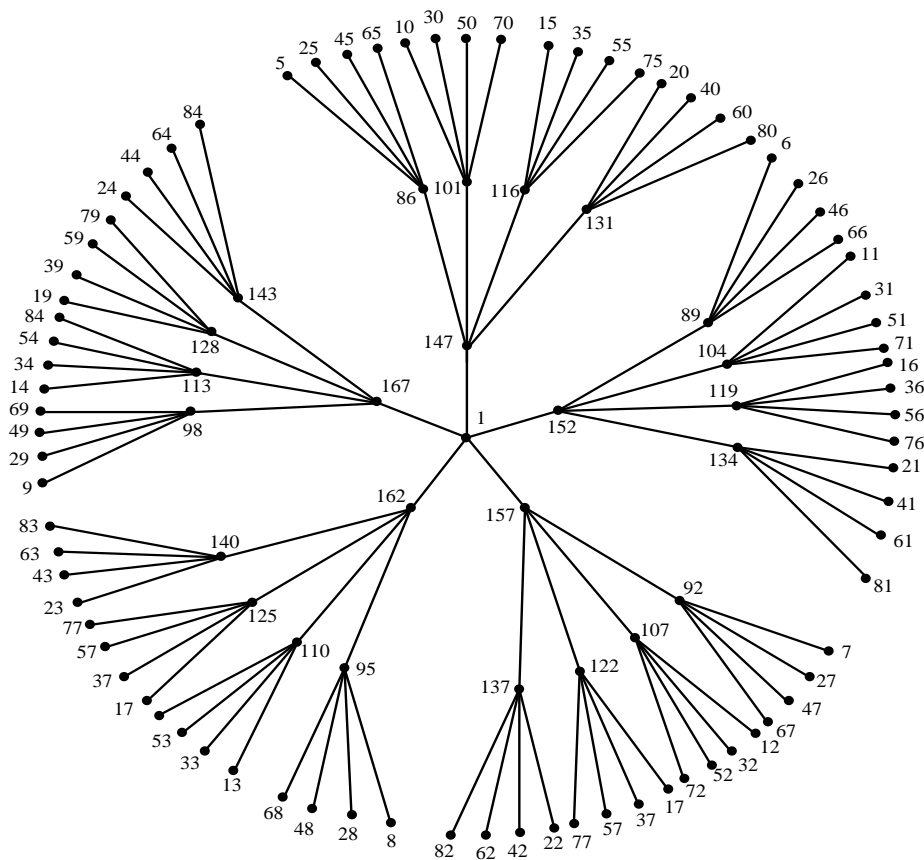


Figure 4. A radio labelling of $T_{3,5}$ which attains the upper bound

Lemma 3.2. For the tree dendrimer graph $T_{n,p}$, the eccentricities $e_0, e_1 \dots e_j$ are given by:

$$e_{i-1} = 2n - i - 1, i = 1, 2 \dots n + 1.$$

Proof: It is obvious that the diameter of the graph is $2n$, and there also exists at least one path $v_1^n, v_1^{n-1} \dots v_1^2, v_1^1, v_1^0, v_1^1, v_1^2 \dots v_{\frac{(p-1)^{n-1}-1}{p-2}}^n$ which passes through the center of the graph. Hence, the consecutive eccentricities are different by exactly 1. Therefore, the eccentricities of $T_{n,p}$ are $e_0, e_1 \dots e_n$. That is, $e_{i-1} = 2n - i - 1, i = 1, 2 \dots n + 1$.

Lemma 3.3. For the tree dendrimer graph $T_{n,p}$, the number of vertices that have eccentricities $e_0, e_1 \dots e_{j=n}$ are given by $m_n = 1$ and $m_{i-1} = p(p-1)^{n-i}, i = 1, 2 \dots n$.

Proof: We know that every pendant vertex in $T_{n,p}$ has $(p-1)$ diametrically opposite vertices. Moreover, the number of pendant vertices in $T_{n,p}$ is $p(p-1)^{n-1}$. Therefore, the number of vertices that have eccentricity $e_0 = 2n$ is $m_0 = p(p-1)^{n-1}$. Similarly, we obtain the number of vertices with eccentricities $e_i, i = 1, 2 \dots n-1$ as $m_i = p(p-1)^{n-i-1}, i = 1, 2 \dots n-1$. Finally, the centre vertex is the only vertex of radius n ; hence, we have: $m_n = 1$ and $m_{i-1} = p(p-1)^{n-i}, i = 1, 2 \dots n$.

e) Theorem 3.6: Let A be a tree dendrimer graph, $T_{n,p}$ ($n, p > 2$), of order $p \left(\frac{(p-1)^{n-1}}{p-2} \right) + 1$. Then, the radio number of A satisfies $rn(A) \geq p \left(\frac{(p-1)^{n-1}}{p-2} \right) + n + \left(\sum_{i=1}^{n-1} 2(2n - (2n - i))p(p-1)^{n-i-1} \right)$.

Proof: From Lemma 3.3, we have: $m_j = 1$; hence, we must apply the second part of the result in Theorem 3.5 as follows:

$$\begin{aligned} rn(A) &\geq m - (d - e_j) - (d - e_{j-1}) + \sum_{i=1}^k 2(d - e_i)m_i \\ &= p \left(\frac{(p-1)^{n-1}}{p-2} \right) + 1 - (2n - n) - (2n - (2n - 1)) + \left(\sum_{i=1}^{n-1} 2(2n - (2n - i))p(p-1)^{n-i-1} \right) + \\ &2(2n - (2n - n)) = p \left(\frac{(p-1)^{n-1}}{p-2} \right) + n + \left(\sum_{i=1}^{n-1} 2(2n - (2n - i))p(p-1)^{n-i-1} \right) \end{aligned}$$

Hence, the theorem is proven.

By combining Theorem 3.4 with Theorem 3.6, the following theorem is obtained:

f) Theorem 3.7: For $p \geq 2n - 3$, the radio number of tree dendrimer graph $T_{n,p}$ ($n, p > 2$) satisfies $p \left(\frac{(p-1)^{n-1}}{p-2} \right) + n + \left(\sum_{i=1}^{n-1} 2(2n - (2n - i))p(p-1)^{n-i-1} \right) \leq rn(T_{n,p}) \leq n + (2n - 1)p + 1 + \sum_{i=1}^{n-1} (2l - 1)p((p-1)^{n-l-1})(p-2) + (2l - 1)p(p)^{n-l-1} - 1$.

4. CONCLUSION

The upper bound for the $L(2,1)$ labelling number of $T_{n,p}$ as $3p + 1$ for $p, n > 2$ has been obtained in this research study. The upper and lower bounds have also been determined for the radio labelling for $T_{n,p}$. For $p < 2n - 3$, the radio labelling problem for $T_{n,p}$ is still an open problem. Further research can be extended to identify more chemical structures and study their properties due to their various applications in the field of communication engineering.




REFERENCES

- [1] J. R. Griggs and R. K. Yeh, "Labelling graphs with a condition at distance 2," *SIAM Journal on Discrete Mathematics*, vol. 5, no. 4, pp. 586-595, 1992, doi: 10.1137/0405048.
- [2] G. Chartrand, D. Erwin, and F. Harary, "Radio Labelling of Graphs," *Bulletin of the Institute of Combinatorics and its Applications*, vol. 33, pp. 77-85, 2001.
- [3] D. Fotakis, G. Pantziou, G. Pentaris, and P. Spirakis, "Frequency Assignment in Mobile and Radio Networks," *DIMACS series in Discrete Mathematics and Theoretical Computer Science*, vol. 45, pp.73-90, 1999.
- [4] J. Fiala, P. Golvach, and J. Kratochvíl "Distance Constrained Labelings of Graphs of Bounded Treewidth," *Proceedings of ICALP 2005*, pp. 360-372, 2005, doi: 10.1007/11523468_30.
- [5] F. Havet, M. Klazar, J. Kratochvíl, D. Kratsch, and M. Liedloff, "Exact algorithms for $L(2, 1)$ -labeling of graphs," *INRIA*, July 2008, [Online]. Available: <https://hal.inria.fr/inria-00303330>
- [6] K. J. Szaniawski and P. Rzażewski, "On Improved Exact Algorithms for $L(2,1)$ -Labeling of Graph," In: Iliopoulos C.S., Smyth W.F. (eds) *Combinatorial Algorithms*, "IWOCA 2010. Lecture Notes in Computer Science", vol. 6460, 2011, pp. 34-37, doi: 10.1007/978-3-642-19222-7_4.
- [7] G. Chang and D. Kuo, "The $L(2, 1)$ -labeling problem on graphs," *SIAM Journal on Discrete Mathematics*, vol. 9, no. 2, pp. 309-316, 1996, doi: 10.1137/S0895480193245339.
- [8] D. Goncalves, "On the $L(p,1)$ -labelling of graphs," *Discrete Mathematics*, vol. 308, pp. 1405-1414, 2008, doi: 10.1016/j.disc.2007.07.075.
- [9] H. L. Bodlaender, T. Kloks, R. B. Tan, and J. Van Leeuwen, "Approximations for λ colorings of graphs," *The Computer Journal*, vol. 47, no. 2, pp. 193-204, 2004, doi: 10.1093/comjnl/47.2.193.
- [10] D. Sakai, "Labeling chordal graphs: distance two conditions," *SIAM Journal on Discrete Mathematics*, vol. 7, no. 1, pp. 33-140, 1994, doi: 10.1137/S0895480191223178.
- [11] K. M. B. Smitha and K. Thirusangu, "Distance Two Labeling of Quadrilateral Snake Families," *International Journal of Pure and Applied Mathematical Sciences*, vol. 9, no. 2, pp. 283-298, 2016, [Online]. Available: https://www.ripublication.com/ijpams16/ijpamsv9n2_19.pdf
- [12] C. Kujur, D. A. Xavier, S. A. Raja, and F. Xavier, " $L(2,1)$ -Labeling for Bloom Graph," *International Journal of Mathematics and its Applications*, vol. 5, no. 4, pp. 437-447, 2017, [Online]. Available: <http://ijmaa.in/v5n4-d/437-447.pdf>
- [13] K. Yenoke, D. F. Xavier, and T. Maryson, " $L(2,1)$ -Labeling of Oxide and Silicate Networks," *Journal of Computer and Mathematical Sciences*, vol. 11, no. 6, pp. 27-33, 2020.




- [14] S. K. Vaidya and D. D. Bantva, "Radio Number for Total Graph of Paths," *ISRN Combinatorics*, vol. 2013, pp. 1-5, 2013, doi: 10.1155/2013/326038.
- [15] R. Cada, J. Ekstein, P. Holub, and O. Togni, "Radio labelling of distance graphs," *Discrete Applied Mathematics*, vol. 161, no. 18, pp. 2876-2884, 2013, doi: 10.1016/j.dam.2013.06.024.
- [16] D. D. F. Liu, "Radio number for trees," *Discrete Mathematics*, vol. 308, pp. 1153-1164, 2008, doi: 10.1016/j.disc.2007.03.066.
- [17] B. M. Kim, W. Hwang, and B. C. Song, "Radio number for the product of a path and a complete graph," *Journal of Combinatorial Optimization*, vol. 30, no. 1, pp. 139-149, 2015, doi: 10.1007/s10878-013-9639-3.
- [18] R. Bharati and K. Yenoke, "On the radio number of hexagonal mesh," *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 79, pp. 235-244, 2011.
- [19] D. Bantva, "A Lower Bound for the Radio Number of Graphs," *Conference on Algorithms and Discrete Applied Mathematics*, 2019, pp. 161-173, doi: 10.1007/978-3-030-11509-8_14.
- [20] S. Das, S. C. Ghosh, S. Nandi, and S. Sen, "A lower bound technique for radio k -coloring," *Discrete Mathematics*, vol. 340, no. 5, pp. 855-861, 2017, doi: 10.1016/j.disc.2016.12.021.
- [21] K. Yenoke, P. Selvagopal, K. M. B. Smitha, and R. Cranston, "Bounds for the Radio Number of Mesh derived Architecture," *International Journal of Innovative Science, Engineering and Technology*, vol. 9, no. 6, pp. 4806-4810, 2020, doi: 10.15680/IJRSET.2020.0906005.
- [22] M. B. Ahmadi and M. Sadeghimehr, "Second-order connectivity index of an infinite class of dendrimer nanostars," *Digest Journal of Nanomaterials and Biostructures*, vol. 4, no. 4, pp. 639-643, 2009.
- [23] M. K. Siddique, M. Imran, and A. Ahmad, "On Zagreb indices, Zagreb polynomials of some nanostar dendrimers," *Applied Mathematics and Computation*, vol. 280, pp. 132-139, 2016, doi: 10.1016/j.amc.2016.01.041.
- [24] J. Yang and F. Xia, "The Eccentric connectivity index of dendrimers," *International Journal of Contemporary Mathematical Sciences*, vol. 5, no. 45, pp. 2231-2236, 2010, [Online]. Available: <http://www.m-hikari.com/ijcms-2010/45-48-2010/yangIJCMS45-48-2010.pdf>
- [25] B. Rajan, I. Rajasingh, K. Yenoke, and P. Manuel, "Radio number of graphs with small diameter," *International Journal of Mathematics and Computer Science*, vol. 2, no. 3, pp. 209-220, 2007.

BIOGRAPHIES OF AUTHORS



Kins Yenoke    received his UG and PG degrees from Manonmaniyam Sundaranar University, Tirunelveli, Tamil Nadu, India. He received his M.Phil. and Ph.D. in Mathematics from Loyola College, Chennai, India. He is currently working as an Assistant Professor in the department of Loyola College, Chennai, India. His areas of research in graph theory are channel assignment problems and various graph labelling problems. Along with 14 years of teaching experience, he has played key roles in organizing many international conferences, seminars, and workshops. His passion involves motivating the young minds to excel in the current research areas and encouraging the staff members to write research articles through faculty development programs. He can be contacted through this email: jecinthokins@rediffmail.com



Mohammed K. A. Kaabar    received all his undergraduate and graduate degrees in Mathematics from Washington State University (WSU), Pullman, WA, USA. Kaabar has a global diverse experience in teaching and has worked as adjunct math Professor, math lab instructor, and lecturer at various US institutions such as Moreno Valley College, Washington State University, and Colorado Early Colleges. He authored two math textbooks, and he is an invited referee for more than 200 Science, Technology, Engineering, and Mathematics (STEM) international conferences all over the world. He served as an editor for the American Mathematical Society (AMS) Graduate Student Blog, and he is also a full editor for an educational program (Mathematics Section) at California State University, Long Beach, CA, USA. He is an approved member of Science and Democracy Network (SDN) which is a part of the Science, Technology, and Society program at Harvard Kennedy School. Kaabar is also currently serving as an editor for 17 international scientific journals in applied mathematics and engineering. He is also a member of the expert editorial advisory board for the Big Data and Advanced Wireless Technologies Springer Book Series. His research interests are fractional calculus, applied mathematics, mathematical physics, and math education. He can be contacted through this email: mohammed.kaabar@wsu.edu