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Energy analysis and comparative study of n-wheel graphs in hierarchical wireless sensor network architectures

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ABSTRACT

The energy analysis of the newly introduced n-wheel graph, employs diverse matrix representations such as the adjacency matrix, Laplacian matrix, and maximum degree matrix. This novel graph model resembles a hierarchical wireless sensor network (WSN), with a central hub serving as the communication center. The graph is organized into cycles, reflecting tiers of devices or sensors, with the hub managing wireless communication across these tiers. Through comparative analysis of energy variations, particularly focusing on ordinary energy, Laplacian energy, and maximum degree energy, offers a deeper understanding on the potential benefits of the n-wheel graph model, guiding future research and practical applications in the design of advanced hierarchical network structures.

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1. INTRODUCTION

This wireless sensor network (WSN) features a hierarchical architecture with a central hub as communication nexus. The network is divided into concentric cycles, starting with a primary cycle of devices like laptops and mobiles near the hub, and extending outward to include devices at increasing distances. The study of WSNs appears in numerous papers [1]-[3]. The hub manages wireless communication across these cycles, facilitating data aggregation and coordination, thus optimizing network operation as shown in Figure 1. This model resembles a multilevel wheel graph or a n- wheel in graph theory. The analysis of the higher extremities of hierarchical wheel networks is the primary finding of this paper. For basic terminologies and notation [4], [5]. The concept of graph energy was introduced by Ivan Gutman and has its roots in chemistry, stemming from the importance of the total π-electron energy in carbon-based compounds. This has led to various graph energies. Recently, a survey on these graph energies was conducted by Kumar *et al.* [6]. This concept has been widely discussed in the literature; see, for example, many research papers [7]-[9]. Ali *et al.* [10] investigated the metric dimension of certain connected networks, In 2019, Jia-Bao Liu *et al.* [11] determined the generalized wheel networks $(W_{n,m})$'s distance and neighboring energies. Lazaro and Rosario [12] determined the precise upper and lower limits for the connected partition dimension of truncated wheel graphs. In 2022, Vivik *et al.* [13] constructed the Cartesian product of P_m and the double wheel graph DW_n, exploring their associated energy metrics in detail. Kandris *et al.* [14] in 2020 classified several types of WSN applications, focusing on advancements in applications, internal platforms, communication protocols, and network services, also found in many papers, see [15]-[18]. In 2022, Bose *et al.* [19] addressed the localization problem in WSNs, focusing on determining node positions in an arbitrarily graph network.

In this paper we analyze the various forms of graph energy ordinary, Laplacian, and maximum degree energy. These energy metrics correspond to communication costs and network robustness, making them essential for optimizing sensor networks. Our findings show that the hierarchical n-wheel graph outperforms other models in energy efficiency, offering a scalable and reproducible framework for improving WSN architectures.

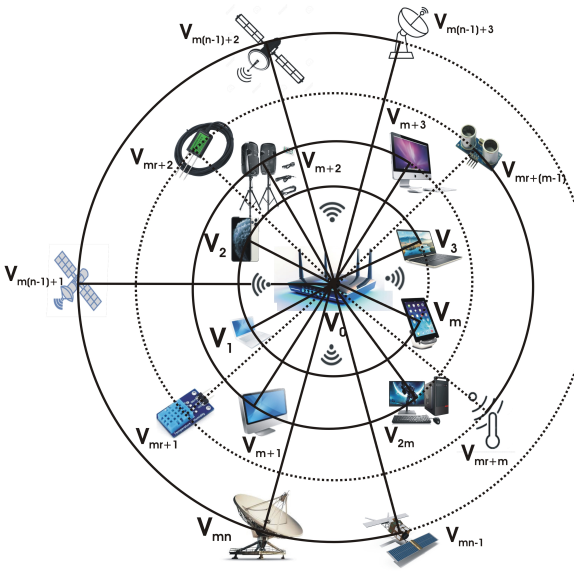

Figure 1. WSN

2. METHOD

The graph is derived from the ordinary wheel network, where all its points are connected to a central hub. This concept is extended hierarchically, iterating the wheel graph structure n times. Analyzing the energy of such a graph presents a novel perspective in graph theory. The refinement of limits for graph energy across diverse graphs is a contemporary approach gaining traction. In this work, we discuss and illustrate the upper limits for the graph spectrum energy, energy of the Laplacian matrix, and energy of the degree matrix of the n-wheel graph. To achieve this, we apply the Cauchy-Schwarz inequality alongside the maxima and minima of higher-order derivatives. Furthermore, we compare the variations in these energies, providing a comprehensive analysis of their energy limits.

Wheel graphs and energy of graphs:

a. Definition

1) Chai *et al.* [20] a central node v is connected to all $\mathsf{m}-1$ nodes of the cycle graph $C_{\mathsf{m}-1}$ to form the wheel graph W_{m}, which has m nodes for $\mathsf{m} \geq 4$.
2) Chai *et al.* [20] two cycles of size m ($2C_{\mathsf{m}}$) connected to a single hub node (K_1) make up a double-wheel graph DW_{m}. All of the cycle nodes link to the hub.
3) Liu *et al.* [11] an n-wheel graph $\mathsf{n}W_{\mathsf{m}}$ of order $\mathsf{k}+1$ comprises n cycles of size m ($\mathsf{n}C_{\mathsf{m}}$) integrated with a central hub node (K_1), where all cycle nodes are interconnected through the hub as shown in Figure 2.
4) (Energy) Ramane *et al.* [9] if a graph G has n nodes and m lines, then the connectivity matrix $A(G)$ is defined as a $\mathsf{n} \times \mathsf{n}$ matrix, where the entry a_{ij} is given a value of 1 if a line links nodes i and j, and null otherwise. $E(G) = \sum_{i=1}^{\mathsf{n}} |\lambda_i|$, where λ_i indicates the characteristic values of the matrix, is the formula used to determine the graph's energy, $E(G)$, which is the sum of the absolute values of the characteristic values of $A(G)$.
5) (Laplacian energy) Barberler [7] with n nodes and m lines of a graph G. The Laplacian matrix $L(G)$ is a $\mathsf{n} \times \mathsf{n}$ matrix in which the degree of node i is $l_{ii} = d_i$, $l_{ij} = -1$ if nodes i and j are not adjacent, and $l_{ij} = null$ if they are not. $L(G) = \sum_{i=1}^{\mathsf{n}} \left|\text{characteristic value}_i - \frac{2\mathsf{m}}{\mathsf{n}}\right|$ is the graph's Laplacian energy.

6) (Maximum degree energy) Adiga and Smitha [8] let G be a simple graph with n nodes, where the degree of node i is indicated by d_i. With $d_{ij} = \max\{d_i, d_j\}$ if nodes i and j are nearby, and $d_{ij} = 0$ otherwise, the maximum degree matrix $M(G)$ is a $\mathsf{n} \times \mathsf{n}$ matrix. $E_{\mathsf{M}}(G) = \sum_{i=1}^{\mathsf{n}} |\mu_i|$ yields the graph's maximum degree energy.
7) (Color energy) Sarathy and Sankar [21] consider a graph G with n nodes and lines m. The color matrix $A_c(G)$ is an $\mathsf{n} \times \mathsf{n}$ matrix where $a_{ij} = 1$ is assigned if nodes i and j are adjacent and have distinct colors, $a_{ij} = -1$ is assigned if they are not adjacent but share the same color, and $a_{ij} = 0$ is assigned otherwise.

b. Theorem

8) Kumar *et al.* [6] the following inequality is true for a graph G with n nodes and m lines: $E(G) \leq \frac{2m}{n} + \sqrt{(n-1)\left[2m - \left(\frac{2m}{n}\right)^2\right]}$, while for a k-regular graph G,$E(G) \leq k + \sqrt{k(n-1)(n-k)}$.
9) Balakrishnan [22] a k-regular G of order n with $k < n-1$ and $\frac{E(G)}{k+\sqrt{k(n-1)(n-k)}} < \epsilon$ exists for every $\epsilon > 0$.
10) Nikivorov [23] if A is an $m \times n$ non-negative matrix with $m \leq n$, and the largest entry in A is α, then: $\varepsilon(A) \leq \alpha \cdot \frac{(m+\sqrt{m})\sqrt{n}}{2}$.

c. Proposition

11) Varlıoğlu and Büyükköse [24] let's G be a connected graph of order n and m lines, such that $G \ncong K_n$. Then $LE(G) < \frac{2m}{n} + \sqrt{m(n^2 - n - 2m) + \left(\frac{2m}{n}\right)^2}$.

d. Theorem

12) Varlıoğlu and Büyükköse [24] let's G be a connected graph of order n and m lines, such that $G \ncong K_n$. Then $LE(G) < \frac{2m}{n} + \sqrt{m(n^2 - n - 2m) + \left(\frac{2m}{n}\right)^2}$.
13) Adiga and Smitha [8] if the characteristic values of the maximum degree of a G are $\mu_1, \mu_2, \ldots, \mu_n$, then: $\sum_{i=1}^{n} \mu_i^2 = -2c_2$.
14) Adiga and Smitha [8] any largest degree characteristic value μ_j for a graph G of order n is $|\mu_j| \leq (n-1)^2$.
15) Sridara *et al.* [25] let G be a graph with $n \geq 3$ nodes and m lines. If $n^2 \geq 4m$, then $\varepsilon(G) \leq \frac{2m}{n} + \sqrt{\frac{2m}{n}} + \sqrt{(n-2)(2m - \frac{2m}{n} - \frac{4m^2}{n^2}})$. Equality holds if and only if G is $\frac{n}{2}K_2$. In the following section, the dissimilar graph energy and its limit of $\mathsf{n}W_{\mathsf{m}}$ are determined.

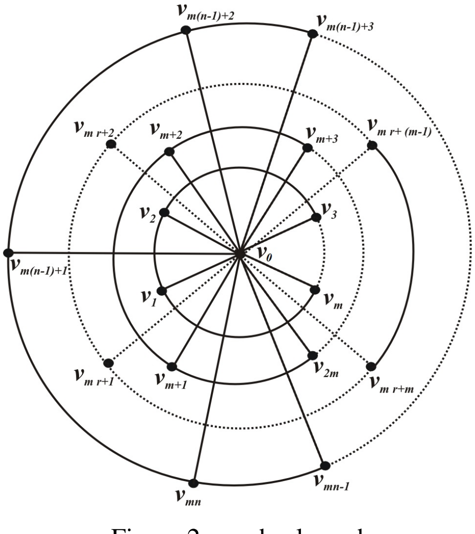

Figure 2. n-wheel graph

3. UPPER LIMITS OF VARIOUS ENERGIES ON n-WHEEL GRAPH

3.1. Theorem 3.1.

Let $\mathsf{n}W_\mathsf{m}$ be a n-wheel graph of n cycles with $\mathsf{m} \geq 4$ nodes and m lines on each cycle then,

$$E(G) \leq \begin{cases} \frac{3\mathcal{K}}{2} + \frac{\mathsf{m}}{2} + 1, \text{ if } \mathsf{m} \equiv 1(\text{mod}2) \text{ and } \mathsf{n} \leq \mathsf{m}; \text{ Also if } \mathsf{m} \equiv 0(\text{mod}2) \text{ and } \mathsf{n} \leq \mathsf{m}+1 \\ \frac{3\mathcal{K}}{2} + \frac{\mathsf{n}}{2}, \text{ if } \mathsf{m} \equiv 1(\text{mod}2) \text{ and } \mathsf{n} > \mathsf{m}; \text{ Also if } \mathsf{m} \equiv 0(\text{mod}2) \text{ and } \mathsf{n} > \mathsf{m}+1. \end{cases}$$

Proof. The matrix connection of the n-wheel graph is expressed as:

$$A(nW_m) = \begin{cases} 1, & \text{if } i \text{ and } j \text{ are connected,} \\ 0, & \text{if } i \text{ and } j \text{ are not connected.} \end{cases}$$

The connection matrix of the n-wheel graph is represented with 1's, where $\mathcal{K} = mn.$ $i = 1,$ j and i vary from 2 to $\mathcal{K}+1,$ $j = 1,$ $i = (\mathsf{n}-1)\mathsf{m}+2,$ $j = \mathcal{K}+1,$ $i = \mathcal{K}+1,$ $j = (\mathsf{n}-1)\mathsf{m}+2,$ $(\mathsf{n}-1)\mathsf{m}+2 \leq i \leq \mathcal{K},$ $j = i+1,$ $(\mathsf{n}-1)\mathsf{m}+3 \leq i \leq \mathcal{K}+1,$ $j = i-1.$

Non-connections are marked by 0's at diagonal and other non-connection positions. Here, n is the number of cycles, and m is the number of nodes per cycle. The n-wheel graph is represented by its connection matrix as:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_\mathsf{m} \\ v_{\mathsf{m}+1} \\ v_{\mathsf{m}+2} \\ v_{\mathsf{m}+3} \\ \vdots \\ v_\mathcal{K} \\ v_{\mathcal{K}+1} \end{array} \begin{array}{c} \begin{array}{ccccccccccc} v_1 & v_2 & v_3 & \cdots & v_\mathsf{m} & v_{\mathsf{m}+1} & v_{\mathsf{m}+2} & v_{\mathsf{m}+3} & \cdots & v_\mathcal{K} & v_{\mathcal{K}+1} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 1 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \end{array}$$

The characteristic equation of the connection matrix of order $\mathcal{K}+1$ is formed by setting the determinant of $\det(A(nW_\mathsf{m}) - \lambda I)$ to 0. With exactly $\mathcal{K}+1$ roots, this equation has the form $(-\lambda)^{\mathcal{K}+1} + \text{trace}(-\lambda)^{\mathcal{K}} + \ldots + \text{determinant}(A) = 0$. As a result, $\mathcal{K}+1$ characteristic values exist. i.e, $\lambda_1, \lambda_2, \ldots, \lambda_{\mathcal{K}+1}$. Also $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{\mathcal{K}+1}$. The energy $E = \sum_{i=1}^{\mathcal{K}+1} |\lambda_i|$. It is evident that for the n-wheel graph $E_1 < E_2 < \ldots < E_{\mathcal{K}+1}$. By Cauchy Schwarz inequality $\left(\sum_{i=1}^{\mathcal{K}+1} |\lambda_i|\right)^2 \leq \sum_{i=1}^{\mathcal{K}+1} |1| \sum_{i=1}^{\mathcal{K}+1} |\lambda_i|^2$

$$\left(\sum_{i=2}^{k} |\lambda_i| - |\lambda_1| - |\lambda_{\mathcal{K}+1}|\right)^2 \leq \left(\sum_{i=2}^{\mathcal{K}} |1| - 2\right) \left(\sum_{i=2}^{\mathcal{K}} |\lambda_i|^2 - |\lambda_1|^2 - |\lambda_{\mathcal{K}+1}|^2\right)$$

$$\sum_{i=2}^{\mathcal{K}} |\lambda_i| \leq |\lambda_1| + |\lambda_{\mathcal{K}+1}| + \sqrt{(\mathcal{K}-2) \left(\sum_{i=2}^{k} |\lambda_i|^2 - |\lambda_1|^2 - |\lambda_{\mathcal{K}+1}|^2\right)}$$

$$\frac{1}{\sqrt{\mathcal{K}}} E(G) \leq \frac{1}{\sqrt{\mathcal{K}}} (|\lambda_1| + |\lambda_{\mathcal{K}+1}|) + \sqrt{(\mathcal{K}-2) \left(\sum_{i=2}^{\mathcal{K}} |\lambda_i|^2 - |\lambda_1|^2 - |\lambda_{\mathcal{K}+1}|^2\right)}$$

Now let $|\lambda_1| = x$ and $|\lambda_{\mathcal{K}+1}| = y$. $\frac{1}{\sqrt{\mathcal{K}}} [E(G)] \leq \frac{1}{\sqrt{\mathcal{K}}} \left[x + y + \sqrt{(\mathcal{K}-2) \left(\sum_{i=2}^{\mathcal{K}} |\lambda_i|^2 - x^2 - y^2\right)}\right]$

Case 1: $m \equiv 1(\text{mod}2)$ and $n \leq m$

Consider the function $f(x, y) = \frac{1}{\sqrt{\mathcal{K}}} \left[x + y + \sqrt{(\mathcal{K}-2) \left\{\left(\frac{3\mathcal{K}}{2} + \frac{m}{2} + 1\right)^2 - x^2 - y^2\right\}}\right]$

Differentiating $f(x, y)$ partially up to the second order derivative with respect to x and y,

$f_x = \frac{1}{\sqrt{\mathcal{K}}} - \frac{x(\mathcal{K}-2)}{\sqrt{\mathcal{K}}\sqrt{(\mathcal{K}-2)\{\left(\frac{3\mathcal{K}}{2} + \frac{m}{2} + 1\right)^2 - x^2 - y^2\}}}$, $f_y = \frac{1}{\sqrt{\mathcal{K}}} - \frac{y(\mathcal{K}-2)}{\sqrt{\mathcal{K}}\sqrt{(\mathcal{K}-2)\{\left(\frac{3\mathcal{K}}{2} + \frac{m}{2} + 1\right)^2 - x^2 - y^2\}}}$,

$f_{xx}=-\frac{\sqrt{\mathcal{K}-2}\left[\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)^2-y^2\right]}{\sqrt{\mathcal{K}}\left[\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)^2-x^2-y^2\right]^{\frac{3}{2}}}$, $f_{yy}=-\frac{\sqrt{\mathcal{K}-2}\left[\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)^2-x^2\right]}{\sqrt{\mathcal{K}}\left[\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)^2-x^2-y^2\right]^{\frac{3}{2}}}$ and
$f_{xy}=-\frac{xy\sqrt{\mathcal{K}-2}}{\sqrt{\mathcal{K}}\left[\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)^2-x^2-y^2\right]^{\frac{3}{2}}}$.
set $f_x=0$ and $f_y=0$, which leads to the determine the maxima or minima of the function, $x^2(k-1)+y^2=\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)^2$ and $x^2+(\mathcal{K}-1)y^2=\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)^2$.
The stationary points obtained by solving the above equations are $x=y=\frac{1}{\sqrt{\mathcal{K}}}\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)$. At this point the values are $f_{xx}=f_{yy}=-\frac{\mathcal{K}-1}{(\mathcal{K}-2)\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)}\leq 0$, $f_{xy}=-\frac{1}{(mn-2)\left(\frac{3mn}{2}+\frac{m}{2}+1\right)}\leq 0$ and $\Delta=f_{xx}.f_{yy}-(f_{xy})^2=\frac{\mathcal{K}}{(\mathcal{K}-2)\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)^2}\geq 0$. As a result, $f(x,y)$ reaches its largest value at $x=y=\frac{1}{\sqrt{\mathcal{K}}}\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)$.
By far the function's largest metrics is $f\left(\frac{1}{\sqrt{\mathcal{K}}}\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right),\frac{1}{\sqrt{\mathcal{K}}}\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)\right)$

$$=\frac{1}{\sqrt{\mathcal{K}}}\left[\frac{2}{\sqrt{\mathcal{K}}}\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)\right]+\frac{1}{\sqrt{\mathcal{K}}}\sqrt{(\mathcal{K}-2)\left\{\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)^2-\frac{2}{\mathcal{K}}\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)^2\right\}}$$

$$=\frac{1}{\sqrt{\mathcal{K}}}\left[\frac{2}{\sqrt{\mathcal{K}}}\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)+\sqrt{\frac{(\mathcal{K}-2)^2\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)^2}{\mathcal{K}}}\right]=\frac{1}{\mathcal{K}}\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)[2+(\mathcal{K}-2)]$$

Thus $f\left(\frac{1}{\sqrt{\mathcal{K}}}\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right),\frac{1}{\sqrt{\mathcal{K}}}\left(\frac{3\mathcal{K}}{2}+\frac{m}{2}+1\right)\right)\leq\frac{3\mathcal{K}}{2}+\frac{m}{2}+1$. Hence under this case the energy limit is $E(G)\leq\frac{3\mathcal{K}}{2}+\frac{m}{2}+1$. The ordinary energy and upper limits of different n-wheel graphs are measured using MATLAB programming and tabulated in Table 1 and plotted in Figure 3. This suggests that the energy of the network escalates as the graph expands, depending on the quantity of cycles and the nodes located within those cycles [25]. The proof of the remaining cases in this theorem is similar to **Case 1.** It follows the same method of defining and differentiating the function to attain the maxima. The highest value of the energy limits is found to be under.
Case 2: $m\equiv 1(\text{mod}2)$ and $n>m$ is $E(G)\leq\frac{3\mathcal{K}}{2}+\frac{n}{2}$,
Case 3: $m\equiv 0(\text{mod}2)$ and $n\leq m+1$: $E(G)\leq\frac{3\mathcal{K}}{2}+\frac{m}{2}+1$,
Case 4: $m\equiv 0(\text{mod}2)$ and $n>m+1$ is $E(G)\leq\frac{3\mathcal{K}}{2}+\frac{n}{2}$.
Illustration: Table 1 shows the ordinary energy and limits of the n-wheel graph as shown in Figure 3.

Table 1. n-wheel graph's energy and limit metrics

Graphs	Nodes	Lines	Cycles	Energy	Energy limit
nW_m	$\mathcal{K}+1$	$\mathcal{K}$	n	ε	E
$4W_4$	17	32	4	22.2462	28
$7W_5$	36	70	7	55.3050	56
$12W_7$	85	168	12	124.2941	132
$10W_{10}$	101	200	10	147.5425	157
$20W_{10}$	201	400	20	285.2403	310
$15W_{15}$	226	450	15	315.0698	346
$14W_{17}$	238	476	14	332.3819	366.5
$20W_{20}$	401	800	20	543.1501	612
$16W_{24}$	385	768	16	523.3711	590
$25W_{25}$	626	1250	25	844.3385	951

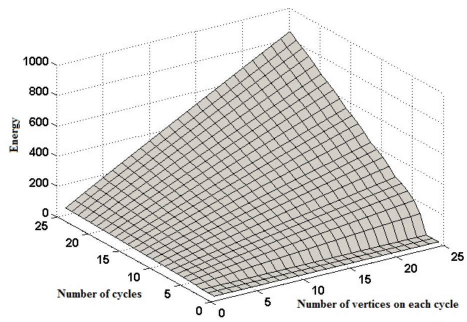

Figure 3. The n-wheel graph's energy metrics

3.2. Theorem 3.2.

Let nW_m be a n-wheel graph of $n\geq 2$ cycles and $m\geq 4$ nodes, with m lines on each cycle then $LE(G)=\mathcal{K}(\mathcal{K}-3)$.
Proof. The procedure for the n-wheel graph's Laplacian matrix is:

$$L(nW_m)=\begin{cases}-1,\ if\ i \text{ and } j \text{ are connected}\\ 0,\ if\ i \text{ and } j \text{ are non-connected}\\ \mathcal{K},\ if\ i=j.\end{cases}$$

n indicates the number of cycles in this graph, while m indicates the number of nodes in each cycle. Thus,

$q = \mathcal{K}+1$ nodes and $p = 2\mathcal{K}$ lines make up the entire graph. here "$\mathcal{K} = \mathsf{mn}$". The connection structure of the Laplacian matrix for the n-wheel graph with -1 as: for i = 1, let i, j vary independently from 2 to $\mathcal{K}+1,\ j = 1, i = (\mathsf{r}-1)\mathsf{m}+2,\ j = \mathsf{rm}+1,\ \mathsf{r} = 1, 2, \ldots, \mathsf{n}, i = \mathsf{rm}+1,\ j = (\mathsf{r}-1)\mathsf{m}+2,\ \mathsf{r} = 1, 2, \ldots, \mathsf{n}, (\mathsf{r}-1)\mathsf{m}+2 \leq i \leq \mathsf{rm},\ j = i+1,\ \mathsf{r} = 1, 2, \ldots, \mathsf{n}, (\mathsf{r}-1)\mathsf{m}+3 \leq i \leq \mathsf{rm}+1,\ j = i-1,\ \mathsf{r} = 1, 2, \ldots, \mathsf{n}$. With the exception of the diagonal elements, which contain $\mathcal{K}$, the non-connected entries are all zeros. The Laplacian matrix of the n-wheel graph constructed as:

$$
\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{\mathsf{m}} \\ v_{\mathcal{K}+1} \\ v_{\mathsf{m}+2} \\ v_{\mathsf{m}+3} \\ \vdots \\ v_{\mathcal{K}} \\ v_{\mathcal{K}+1} \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} v_1 & v_2 & v_3 & \cdots & v_{\mathsf{m}} & v_{\mathsf{m}+1} & v_{\mathsf{m}+2} & v_{\mathsf{m}+3} & \cdots & v_{\mathcal{K}} & v_{\mathcal{K}+1} \end{array} \\
\begin{pmatrix}
\mathcal{K} & -1 & -1 & \ldots & -1 & -1 & -1 & -1 & \ldots & -1 & -1 \\
-1 & \mathcal{K} & -1 & \ldots & 0 & -1 & 0 & 0 & \ldots & 0 & 0 \\
-1 & -1 & \mathcal{K} & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & 0 & 0 & \ldots & \mathcal{K} & -1 & 0 & 0 & \ldots & 0 & 0 \\
-1 & -1 & 0 & \ldots & -1 & \mathcal{K} & 0 & 0 & \ldots & 0 & 0 \\
-1 & 0 & 0 & \ldots & 0 & 0 & \mathcal{K} & -1 & \ldots & 0 & 0 \\
-1 & 0 & 0 & \ldots & 0 & 0 & -1 & \mathcal{K} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & \mathcal{K} & -1 \\
-1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & -1 & \mathcal{K}
\end{pmatrix}
\end{array}
$$

Establish that determinant$(L(nW_{\mathsf{m}}) - \mu I) = 0$ of order $\mathcal{K}+1$, the characteristic polynomial of this connected matrix is $(-\mu)^{\mathcal{K}+1}$ + trace$(-\mu)^{\mathcal{K}} + \ldots +$ determinant$(A) = 0$. It has the roots $\mathcal{K}+1$. Hence, the characteristic values are $\mathcal{K}+1$, i.e., $\mu_1, \mu_2, \ldots, \mu_{\mathcal{K}+1}$. Also $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_{\mathcal{K}+1}$. The energy $E = \sum_{i=1}^{\mathcal{K}+1} |\mu_i|$ For n-wheel graph, $E_1 < E_2 < \ldots < E_{\mathcal{K}+1}$. The Laplacian energy and upper limits of various n-wheel graphs are computed in MATLAB, shown in Table 2, and plotted in Figure 4. The energy of the network increases with the graph's size, driven by the number of cycles and the nodes associated with them. [25]. Let the Laplacian limit be:

$$\sum_{i=1}^{\mathcal{K}+1} \left|\mu_i - \frac{2p}{q}\right| = \mathcal{K}(\mathcal{K}-3); \qquad \sum_{i=1}^{\mathcal{K}+1} \left|\mu_i - \frac{2p}{q}\right|^2 = [\mathcal{K}(\mathcal{K}-3)]^2$$

$$\left|\mu_1 - \frac{2p}{q}\right|^2 + \sum_{i=2}^{\mathcal{K}+1} \left|\mu_i - \frac{2p}{q}\right|^2 = [\mathcal{K}(\mathcal{K}-3)]^2; \quad |\mu_1|^2 - \frac{4p}{q}|\mu_1| + \left|\frac{4p}{q}\right|^2 + \sum_{i=2}^{\mathcal{K}+1} \left|\mu_i - \frac{2p}{q}\right|^2 = [\mathcal{K}(\mathcal{K}-3)]^2$$

$$\sum_{i=2}^{\mathcal{K}+1} \left|\mu_i - \frac{2p}{q}\right|^2 = \frac{4p}{q}|\mu_1| - |\mu_1|^2 - \left(\frac{4p}{q}\right)^2 + [\mathcal{K}(\mathcal{K}-3)]^2$$

Hence $LE(G) = \sum_{i=2}^{\mathcal{K}+1} \left|\mu_i - \frac{2p}{q}\right| = \sqrt{\frac{4p}{q}|\mu_1| - |\mu_1|^2 - \left(\frac{4p}{q}\right)^2 + [\mathcal{K}(\mathcal{K}-3)]^2}$ now substituting $|\mu_1| = x$ in the energy function, it becomes $LE(G) = \sqrt{\frac{4p}{q}x - x^2 - \left(\frac{4p}{q}\right)^2 + [\mathcal{K}(\mathcal{K}-3)]^2}$. Considering the above as an optimizing function, $f(x) = \sqrt{\frac{4p}{q}x - x^2 - \left(\frac{4p}{q}\right)^2 + [\mathcal{K}(\mathcal{K}-3)]^2}$. Differentiating successively up to second order with respect to x, it implies
$f'(x) = \frac{\frac{2p}{q} - x}{\sqrt{\frac{4p}{q}x - x^2 - \left(\frac{4p}{q}\right)^2 + [\mathcal{K}(\mathcal{K}-3)]^2}}$ $\quad f''(x) = -\frac{\{\mathcal{K}(\mathcal{K}-3)\}^2}{\left[\frac{4p}{q}x - x^2 - \frac{4p^2}{q^2} + \{\mathcal{K}(\mathcal{K}-3)\}^2\right]^{\frac{3}{2}}}$. Equating $f'(x) = 0$ and solving it the stationary point is obtained as $x = \frac{2p}{q}$ which helps to analyze the maxima or minima of the function. At this point the value of $f''(x) = -\frac{1}{\mathcal{K}(\mathcal{K}-3)} \leq 0$. Therefore, the function $f(x)$ reaches its highest value at $x = \frac{2p}{q}$. The peak value of the function is attained at this point. $f\left(\frac{2p}{q}\right) = \sqrt{\frac{4p}{q}\left(\frac{2p}{q}\right) - \left(\frac{2p}{q}\right)^2 - \frac{4p^2}{q^2} + [\mathcal{K}(\mathcal{K}-3)]^2}$ $= \mathcal{K}(\mathcal{K}-3)$. Hence the Laplacian energy limit is $LE(G) = \mathcal{K}(\mathcal{K}-3)$ (corrected to four decimals).

Illustration: Table 2 displays the Laplacian energy and its limits for the n-wheel graph as shown in Figure 4.

Table 2. The n-wheel graph's Laplacian energy metrics and their limits

Graphs nW_m	Nodes $\mathcal{K}+1$	Lines $\mathcal{K}$	Cycles n	Laplacian energy $L(\varepsilon)$	Energy limits $L(E)$
$3W_5$	16	30	3	180	180
$14W_7$	99	196	14	9310	9310
$11W_8$	89	176	11	7480	7480
$18W_{10}$	181	360	18	31860	31860
$14W_{11}$	155	308	14	23254	23254
$17W_{15}$	256	510	17	64260	64260
$12W_{18}$	217	432	12	46008	46008
$15W_{21}$	316	630	15	98280	98280
$22W_{22}$	485	968	22	232800	232800
$25W_{25}$	626	1250	25	388750	388750

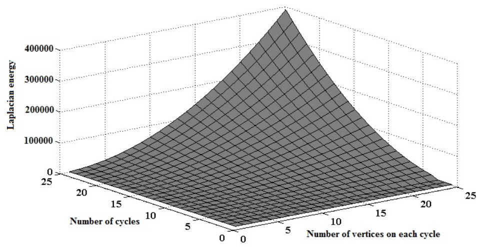

Figure 4. Measures of Laplacian energy on a n-wheel graph

3.3. Theorem 3.3.

Let nW_m be a n-wheel graph of n cycles with $m \geq 4$ nodes and m lines on each cycle then $E_M(G) < \mathcal{K}(m+n+4)$.

Proof. The maximum degree matrix of n - wheel graph is $M(nW_m) = \begin{cases} max\ \{3, \mathcal{K}\},\ if\ i,j \text{ are connected} \\ 0,\ otherwise. \end{cases}$
where "$\mathcal{K} = \mathsf{mn}$" entries at locations $i = 1$, j varies from 2 to $\mathcal{K}+1$ and i vary from 2 to $\mathcal{K}+1$, j = 1 are part of the connected relationships for the n-wheel graph. Furthermore, $i = (\mathsf{n}-1)\mathsf{m}+2, j = \mathcal{K}+1$ contains the value '3'. $i = \mathcal{K}+1, j = (\mathsf{n}-1)\mathsf{m}+2$, along with $(\mathsf{n}-1)\mathsf{m}+2 \leq i \leq \mathcal{K}, j = i+1$, and $(\mathsf{n}-1)\mathsf{m}+3 \leq i \leq \mathcal{K}+1, j = i-1$. Null elements are used to identify non-connected arrangements on the diagonal and else. This is where n and m is the number of cycles & nodes in each cycle. This is the maximum degree matrix of n-wheel graph.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \\ v_{m+1} \\ v_{m+2} \\ v_{m+3} \\ \vdots \\ v_{\mathcal{K}} \\ v_{\mathcal{K}+1} \end{array} \begin{array}{c} \begin{array}{ccccccccccc} v_1 & v_2 & v_3 & \cdots & v_m & v_{m+1} & v_{m+2} & v_{m+3} & \cdots & v_{\mathcal{K}} & v_{\mathcal{K}+1} \end{array} \\ \left(\begin{array}{ccccccccccc} 0 & \mathcal{K} & \mathcal{K} & \ldots & \mathcal{K} & \mathcal{K} & \mathcal{K} & \mathcal{K} & \ldots & \mathcal{K} & \mathcal{K} \\ \mathcal{K} & 0 & 3 & \ldots & 0 & 3 & 0 & 0 & \ldots & 0 & 0 \\ \mathcal{K} & 3 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{K} & 0 & 0 & \ldots & 0 & 3 & 0 & 0 & \ldots & 0 & 0 \\ \mathcal{K} & 3 & 0 & \ldots & 3 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \mathcal{K} & 0 & 0 & \ldots & 0 & 0 & 0 & 3 & \ldots & 0 & 0 \\ \mathcal{K} & 0 & 0 & \ldots & 0 & 0 & 3 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{K} & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 3 \\ \mathcal{K} & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 3 & 0 \end{array}\right) \end{array}$$

It can be observed that if determinant$(M(nW_\mathsf{m}) - \mu I) = 0$, the characteristic polynomial of this connected matrix, with a dimension of $\mathcal{K}+1$, can be expressed as: $(-\mu)^{\mathcal{K}+1}$+trace$(-\mu)^{\mathcal{K}}+\ldots+$determinant$(A) = 0$, which yields exactly $\mathcal{K}+1$ solutions. Consequently, the matrix has $\mathcal{K}+1$ characteristic values, denoted as $\mu_1, \mu_2, \ldots, \mu_{\mathcal{K}+1}$. Furthermore, these characteristic values are arranged in ascending order: $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_{\mathcal{K}+1}$. The energy $E_M(G) = \sum_{i=1}^{\mathcal{K}+1} |\mu_i|$. It is clear that $E_{M_1} < E_{M_2} < \ldots < E_{M_{\mathcal{K}+1}}$. The maximum degree energy and its upper limits of different n-wheel graphs are computed using MATLAB programming and tabulated in Table 3 and plotted in Figure 5. This conveys that the energy of the network increases with the size of the graph in relation to the number of cycles and vertices present on the cycles [25]. To acquire the maximum limits theoretically, the proof is similar to Theorem 3.1.

Illustration: The n-wheel graph's maximum degree energy and energy boundaries are displayed in Table 3. Refer to Figure 5.

Table 3. The n-wheel graph's maximum degree energy metrics and their bounds

Graphs nW_m	Nodes $\mathcal{K}+1$	Lines $\mathcal{K}$	Cycles n	Maximal degree energyε	Energy limit E_M
$10W_4$	41	80	10	620	720
$19W_5$	96	190	19	2214.8	2660
$16W_8$	129	256	16	3353.8	3584
$17W_{11}$	188	374	17	5825.1	5984
$15W_{15}$	226	450	15	7605	7650
$21W_{17}$	358	714	21	14850	14994
$20W_{19}$	381	760	20	16262	16340
$21W_{22}$	463	924	22	21625	21714
$23W_{24}$	553	1104	23	28029	28152
$25W_{25}$	626	1250	25	33633	33750

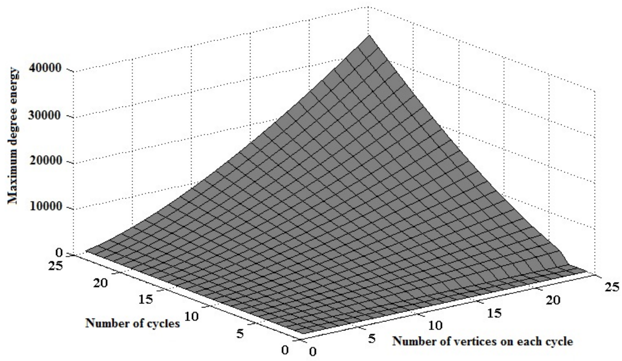

Figure 5. Energy metrics based on the highest degree in the n-wheel graph

3.4. Proposition 3.4.

If $\mu_1, \mu_2, , \ldots, \mu_{2n+1}$ are the characteristic values of $E_c(W_n)$ then $\sum_{i=1}^{2n+1} \mu_i^2 = \delta + \varkappa$.

Proof. The n-wheel color matrix creates two generalized matrix patterns based on the odd and even number of the n-wheel structure.

$$E_c(W_n) = \begin{cases} \kappa_{ij} = 0, & \text{the total of all null entries on the main diagonal where } i = j, \\ \delta_{ij} = 1, & \text{the total of all adjacent nodes with different colors,} \\ \varkappa_{ij} = -1, & \text{the total of all non-adjacent nodes with the same color.} \end{cases}$$

Here $\delta, \varkappa$ denotes the sum of all the elements 1, -1 node colored matrix respectively.

3.5. Theorem 3.5.

Let nW_n be an n-wheel graph consisting of n cycles with $n \geq 4$ nodes and m lines on each cycle. Then $E_c(W_n) < \sqrt{(2n+1)(\delta + \varkappa)}$.

Proof. The color energy upper bound of the n-wheel graph is derived by applying Cauchy-Schwarz inequality and using proposition 3.4.

4. COMPARISON OF ENERGY VARIATIONS OF n-WHEEL GRAPH

After computing the values of ordinary energy, Laplacian energy, and maximum degree energy for an n-wheel graph, an intriguing comparison emerges. Among the three energy measures, the ordinary energy stands out as the lowest, suggesting a relatively uniform distribution of edges throughout the graph. Following closely, the maximum degree energy falls in between, indicating moderate connectivity or centrality of the highest degree nodes. However, the most striking observation arises with the Laplacian energy, which presents a drastically higher value compared to the other energies. This disparity highlights the intricate nature of the graph's structure, possibly indicating the presence of numerous cycles or complex connectivity patterns. Through this comparison, each energy measure unveils distinct facets of the graph's topology, providing valuable insights into its composition and organization. The comparison of these energies is shown in Figure 6.

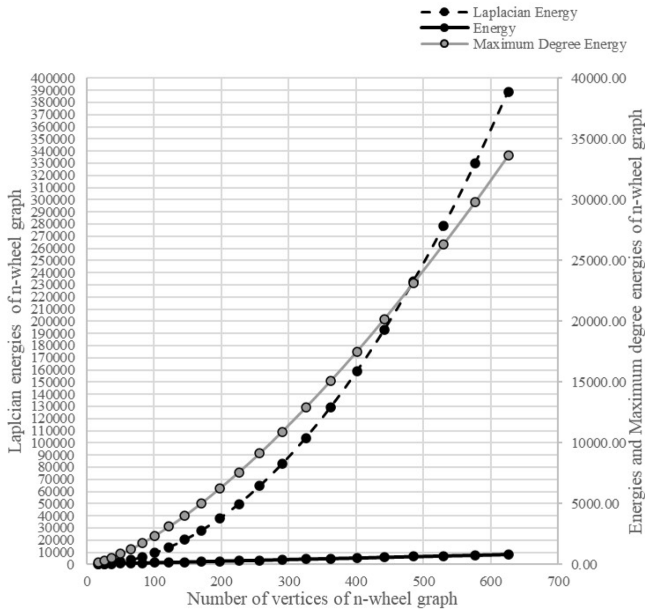

Figure 6. Energy comparison of n-wheel graph

5. CONCLUSION

The concept of energy in graph theory finds extensive applications across diverse fields such as electrical circuits, sensor networks, mathematics, physics, and the chemical sciences. However, due to the inherent complexity of graph structures, establishing generalized bounds on graph energies remains a challenging task. Consequently, numerous researchers have endeavored to refine and enhance these bounds for various types of graphs. This paper primarily explores the analysis of comparison of three distinct energy measures applied to the n-wheel graph. Among these energies, our analysis reveals that the Laplacian energy exhibits the most pronounced influence on the n-wheel graph, surpassing the other two energies examined. Further improved bounds can be achieved for other energy measures of n-wheel graph and other circuit network structures. The potential applications include optimizations in network design, where energy measures inform resilience, as well as in sensor networks and circuit design, where energy stability is paramount. Future work may focus on refining limits for alternative graph structures, supporting broader applications in mathematical modeling and physics.

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C	: **C**onceptualization	I	: **I**nvestigation	Vi	: **Vi**sualization
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CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

DATA AVAILABILITY

Derived data supporting the findings of this study are available from the corresponding author [VVJ] on request.

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