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Quasi-Newton Method for Absolute Value Equation Based on Upper Uniform Smoothing Approximation

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Abstract

In this paper, an upper uniform smooth approximation function of absolute value function is proposed, and some properties of uniform smooth approximation function are studied. Then, absolute value equation (AVE), Ax - |x| = b, where A is a square matrix whose singular values exceed one, is transformed into smooth optimization problem by using the upper uniform smooth approximation function, and solved by quasi-Newton method. Numerical results in solving given AVE problems demonstrated that our algorithm is valid and superior to lower uniform smooth approximation function.

***Keywords**: Quasi-Newton method, absolute value function, absolute value equation, upper uniform smoothing approximation function*

1. Introduction

Consider the AVE:

$$Ax-|x|=b \tag{1}$$

Where $A\in R^{n\times n}$, $x,b\in R^n$, and $|x|$ denotes the vector with absolute value of each component of x. Mangasarian O.L et al. shown that AVE (1) is NP-hard in general form [1].

Mangasarian [2] proposed a finite computational algorithm for AVE, which is solved by a finite succession of linear programs. Semismooth Newton method is proposed for solving the AVE, which largely shortens the computation time than the succession of linear programs (SLP) method In [3, 4]. If the singular values of A exceed 1, then the semismooth Newton iterates are well defined and bounded. However, the global linear convergence of this method is only guaranteed under more stringent condition. In addition, NP-hard *n*-dimensional knapsack feasibility problem is equivalent to AVE [5].

Zhang, et al. proposed a generalized Newton method to AVE, which has global and finite convergence [6]. Louis Caccetta presented a smoothing Newton algorithm to solve the AVE [7], and proved that convergence rate was quadratic. Recently, AVE (1) has been well studied by Jiri Rohn [8-10]. Especially, Jiri gave an method for computing all solutions of AVE. Iterative method, Minimum Norm Solution, and Picard-HSS iteration method to AVE are demonstrated in [11-13]. Harmony search algorithm with chaos and Smooth Newton method based on lower uniform smoothing approximation function are proposed in [14, 15]. Yong has applied AVE to solve two-point boundary value problem of linear differential equation [16].

Compared with literature [15], the basic contribution of present work is that an upper uniform smooth approximation function of absolute value function is presented. After replacing the absolute value function by upper uniform smooth approximation function, the non-smooth AVE is formulated as smooth nonlinear equations, furthermore, an unconstrained smooth optimization problem, and solved by quasi-Newton method.

This paper is outlined as follows. In section 2, upper uniform smoothing approximation function and its properties are studied. In section 3 AVE is formulated as an unconstrained smooth optimization problem. In the same time, some lemmas of AVE are demonstrated. In

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section 4, quasi-Newton method to AVE is given. Numerical examples are considered in section 5. Section 6 concludes the paper.

2. Upper Uniform Smoothing Approximation Function of Absolute Value Function

Definition 2.1 A function $f_\mu := R^1 \to R^1, \mu > 0$ is called a uniformly smoothing approximation function of a non-smooth function $f := R^1 \to R^1$ if, for any $t \in R^1$, f_μ is continuously differentiable, and there exists a constant κ such that:

$$\left| f_\mu(t) - f(t) \right| \le \kappa\mu, \quad \forall \mu > 0,$$

Where $\kappa > 0$ is constant independent on t.

Following we will give a uniformly smoothing approximation function of $\phi(t) = |t|$.

Theorem 2.1 Let $\phi_\mu(t) = \sin\mu \cdot \ln\left(e^{\frac{t}{\sin\mu}} + e^{\frac{-t}{\sin\mu}} \right), 0 < \mu < \frac{\pi}{2}$. Then

(i) $0 < \phi_\mu(t) - \phi(t) \le \sin\mu \ln 2$;

(ii) $\left| \frac{d\left(\phi_\mu(t)\right)}{dt} \right| < 1$, and $\left. \frac{d\left(\phi_\mu(t)\right)}{dt} \right|_{t=0} = 0$.

Proof (i).

$$\phi_\mu(t) - \phi(t) = \sin\mu \ln\left(e^{\frac{t}{\sin\mu}} + e^{\frac{-t}{\sin\mu}} \right) - \sin\mu \ln e^{\frac{|t|}{\sin\mu}} = \sin\mu \ln\left(e^{\frac{t-|t|}{\sin\mu}} + e^{\frac{-t-|t|}{\sin\mu}} \right),$$

Since $|t| = \max\{t, -t\}$, so $t - |t| \le 0, -t - |t| \le 0$. Moreover, $t - |t|$ or $-t - |t|$ is equal to 0 at least. Then $0 = \sin\mu \ln 1 < \sin\mu \ln\left(e^{\frac{t-|t|}{\sin\mu}} + e^{\frac{-t-|t|}{\sin\mu}} \right) \le \sin\mu \ln(1+1) = \sin\mu \ln 2$, That is:

$$0 < \phi_\mu(t) - \phi(t) \le \sin\mu \ln 2.$$

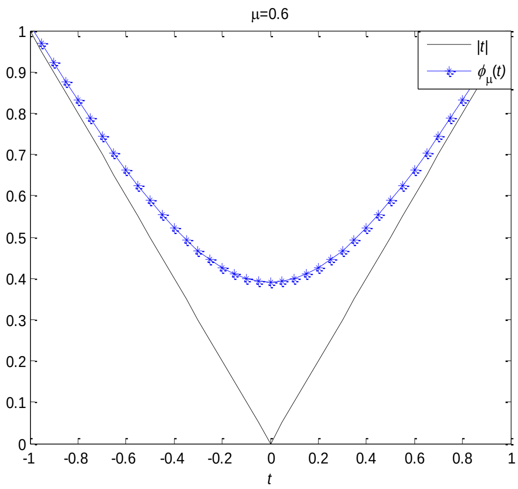
(ii)

$$\frac{d\left(\phi_\mu(t)\right)}{dt} = \frac{e^{\frac{t}{\sin\mu}} - e^{\frac{-t}{\sin\mu}}}{e^{\frac{t}{\sin\mu}} + e^{\frac{-t}{\sin\mu}}} = \frac{e^{\frac{2t}{\sin\mu}} - 1}{e^{\frac{2t}{\sin\mu}} + 1},$$

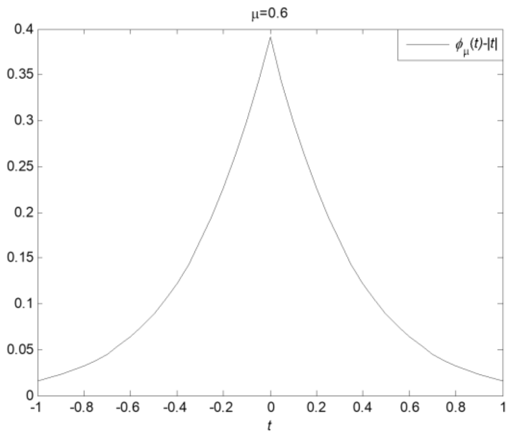
Thus,

$$\left| \frac{d\left(\phi_\mu(t)\right)}{dt} \right| < 1 \text{ and } \left. \frac{d\left(\phi_\mu(t)\right)}{dt} \right|_{t=0} = \left. \left(\frac{e^{\frac{t}{\sin\mu}} - e^{\frac{-t}{\sin\mu}}}{e^{\frac{t}{\sin\mu}} + e^{\frac{-t}{\sin\mu}}} \right) \right|_{t=0} = 0.$$

Figure 1-3 show $\phi_\mu(t)$ and $\phi_\mu(t) - |t|$ with $\mu = 0.6, \mu = 0.2, \mu = 0.06$.

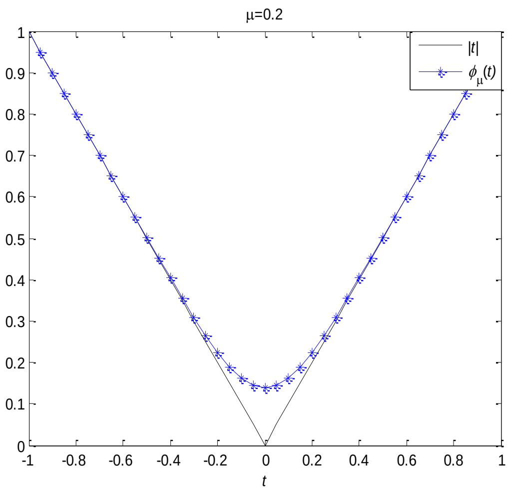


(a) $\phi_\mu(t)$ with $\mu = 0.6$

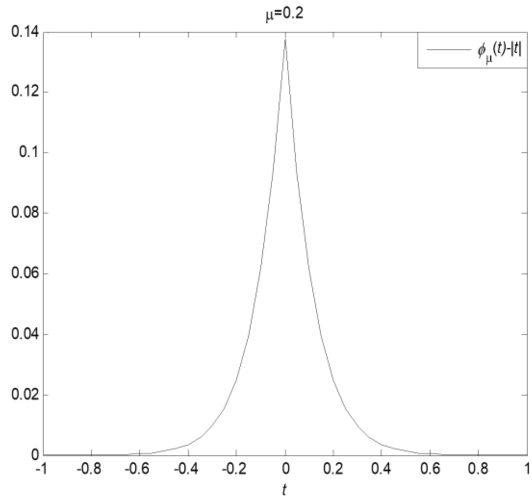


(b) $\phi_\mu(t) - |t|$ with $\mu = 0.6$

Figure 1. $\phi_\mu(t)$ and $\phi_\mu(t) - |t|$ with $\mu = 0.6$

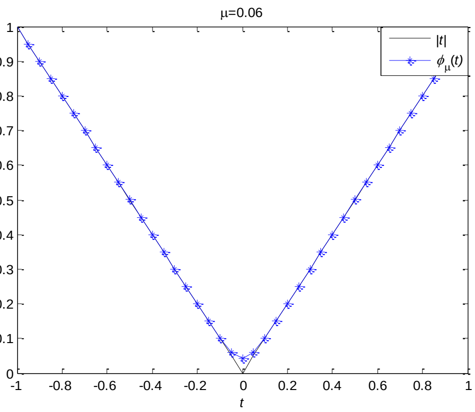


(a) $\phi_\mu(t)$ with $\mu = 0.2$

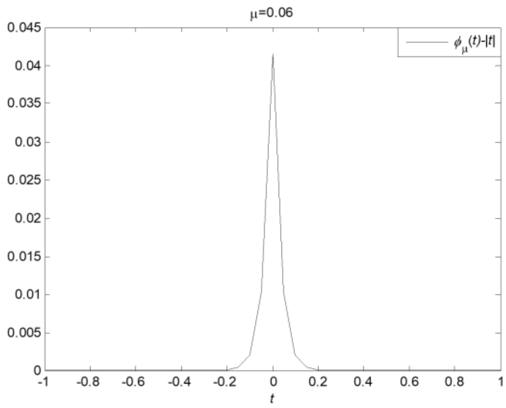


(b) $\phi_\mu(t)-|t|$ with $\mu=0.2$

Figure 2. $\phi_\mu(t)$ and $\phi_\mu(t)-|t|$ with $\mu=0.2$



(a) $\phi_\mu(t)$ with $\mu=0.06$



(b) $\phi_\mu(t)-|t|$ with $\mu=0.06$

Figure 3. $\phi_\mu(t)$ and $\phi_\mu(t)-|t|$ with $\mu=0.06$

From Figure 1-3, with the decrease of parameter μ, $\phi_\mu(t)$ is decreasing, and $\lim_{\mu\to 0^+}\phi_\mu(t)=\phi(t)$. So $\phi_\mu(t)$ is an upper uniform smooth approximation function of absolute value function $\phi(t)=|t|$.

3. Smoothing approximation function of AVE

Define $H := R^n \to R^n$ by:

$$H(x) := Ax - |x| - b. \quad (2)$$

H is a non-smooth function because of the non- smooth of the $\phi(t)=|t|$. In theory, x is a solution to AVE (1) if and only if $H(x)=0$. In this section we give a smoothing function of H and study its properties. Firstly we give some lemmas.

Lemma 3.1 [1]

(a) AVE (1) is uniquely solvable for any $b\in R^n$ if the singular values of A exceed 1.

(b) AVE (1) is uniquely solvable for any $b\in R^n$ if $\|A^{-1}\|<1$.

Lemma 3.2. For a matrix $A\in R^{n\times n}$, the following conditions are equivalent.

(a) The singular values of A exceed 1.

(b) $\|A^{-1}\|<1$.

Lemma 3.3[15]. Suppose that A is nonsingular and $\|A^{-1}B\|<1$. Then, $A+B$ is nonsingular.

Lemma 3.4[15]. Let $D=diag(d)$ with $d_i\in[-1,1], i=1,2,\ldots,n$. Suppose that $\|A^{-1}\|<1$. Then, $A+D$ is nonsingular.

Definition 3.1[15]. A function $H_\mu := R^n \to R^n, \mu>0$ is called a uniformly smoothing approximation function of a non-smooth function $H := R^n \to R^n$ if, for any $x\in R^n$, H_μ is continuously differentiable, and there exists a constant κ such that:

$$\|H_\mu(x)-H(x)\|\le \kappa\mu, \quad \forall \mu>0.$$

Where $\kappa>0$ is constant without depend on x.

Let $\phi(x)=(\phi(x_1),\phi(x_2),\ldots,\phi(x_n))^T$, $\phi(x_i)=|x_i|, i=1,2,\ldots,n$.

For any $0<\mu<\frac{\pi}{2}$, let

$$\phi_\mu(x)=(\phi_\mu(x_1),\phi_\mu(x_2),\ldots,\phi_\mu(x_n))^T,$$

$$\phi_\mu(x_i)=\sin\mu\cdot\ln\left(e^{\frac{x_i}{\sin\mu}}+e^{\frac{-x_i}{\sin\mu}}\right), i=1,2,\ldots,n.$$

Define $H_\mu := R^n \to R^n$ by:

$$H_\mu(x)=Ax-\phi_\mu(x)-b. \quad (3)$$

Clearly, H_μ is a smoothing function of H. Now we give some properties of H_μ, which will be used in the next section.

By simple computation, for any $\mu > 0$, the Jacobian of H_μ at $x \in R^n$ is:

$$H_\mu'(x) = A - diag\left(\phi_\mu'(x_1), \phi_\mu'(x_2), \cdots, \phi_\mu'(x_n)\right)$$

Where,

$$\phi_\mu'(x_i) = \frac{e^{\frac{x_i}{\sin\mu}} - e^{\frac{-x_i}{\sin\mu}}}{e^{\frac{x_i}{\sin\mu}} + e^{\frac{-x_i}{\sin\mu}}}, i = 1, 2, \cdots, n.$$

Theorem 3.1. Suppose that $\|A^{-1}\| < 1$. Then $H_\mu'(x)$ is nonsingular for any $0 < \mu < \frac{\pi}{2}$.

Proof: Note that for any $0 < \mu < \frac{\pi}{2}$, by Theorem 2.1, $\left|\phi_\mu'(x_i)\right| < 1, i = 1, 2, \cdots, n$.
Hence, by Lemma 3.4, we have $H_\mu'(x)$ is nonsingular.

Theorem 3.2. Let $H(x)$ and $H_\mu(x)$ be defined as (2) and (3), respectively. Then, $H_\mu(x)$ is a uniformly smoothing approximation function of $H(x)$.

Proof: For any $0 < \mu < \frac{\pi}{2}$, by Theorem 2.1.

$$\|H_\mu(x) - H(x)\| = \|\phi_\mu(x) - \phi(x)\| = \sqrt{\sum_{i=1}^{n} |\phi_\mu(x_i) - \phi(x_i)|^2} \le \sqrt{n} \ln 2 \cdot \sin\mu.$$

Denote $x(\mu)$ is the solution of (3), then $x(\mu)$ converges to the solution of (1) as μ goes to zero.

For any $0 < \mu < \frac{\pi}{2}$, Define $\theta := R^n \to R$ and $\theta_\mu := R^n \to R$ by:

$$\theta(x) = \frac{1}{2}\|H(x)\|^2, \ \theta_\mu(x) = \frac{1}{2}\|H_\mu(x)\|^2.$$

Theorem 3.3. Suppose that $\|A^{-1}\| < 1$. Then, for any $0 < \mu < \frac{\pi}{2}$ and $x \in R^n$, $\nabla\theta_\mu(x) = 0$ implies that $\theta_\mu(x) = 0$.

Proof: For any $0 < \mu < \frac{\pi}{2}$ and $x \in R^n$, $\nabla\theta_\mu(x) = [H_\mu'(x)]^T H_\mu(x)$. By Theorem 3.1, $H_\mu'(x)$ is nonsingular. Hence, if $\nabla\theta_\mu(x) = 0$, then $H_\mu(x)$ and $\theta_\mu(x) = 0$.

4. Quasi-Newton Method

Following we give quasi-Newton method for solving $H_\mu(x) = 0$.

Algorithm 4.1. Quasi-Newton method for AVE.

Step 1. Given $0 < \mu_0 < \frac{\pi}{2}, k = 0$. Establish the objective function.

$$\theta_{\mu_k}(x) = \frac{1}{2}\left\| H_{\mu_k}(x) \right\|^2 .$$

Step 2. Apply quasi-Newton method to solve $\min_x \theta_{\mu_k}(x)$. Let $x_k = \arg\min_x \theta_{\mu_k}(x)$.

Step 3. Check whether the stopping rule is satisfied. If satisfied, stop.

Step 4. Let $\mu_{k+1} = \mu_k + (1 - e^{\mu_k}) / e^{\mu_k}$, $k := k+1$. Return to step 2.

5. Numerical Results

Some numerical tests are performed in order to illustrate the efficiency of the quasi-Newton method to AVE. We set $\mu_0 = \sin 1$, and initial point $x^0 = (10, 10, \cdots, 10)^T$. All experiments were performed on MatlabR2009a with Intel(R) Core(TM) 4×3.3GHz and 2GB RAM.

5.1. AVE Problems

Problem 1. Let A be a matrix whose diagonal elements are 500 and the nondiagonal elements are chosen randomly from the interval $[1, 2]$ such that A is symmetric. Let $b = (A - I)e$ such that $x = (1, 1, \cdots, 1)^T$ is the unique solution.

Problem 2. Let matrix A is given by:

$$a_{ii} = 4n, \quad a_{i,i+1} = a_{i+1,i} = n, \quad a_{ij} = 0.5, \quad i = 1, 2, \cdots, n.$$

Let $b = (A - I)e$ such that $x = (1, 1, \cdots, 1)^T$ is unique solution.

Problem 3. Consider AVE problems with singular values of A exceeding 1 where the data (*A, b*) are generated by the Matlab scripts:

```
rand('state',0);R=rand(n,n);
A=R'*R+n*eye(n,n);b=(A-eye(n,n))*ones(n,1);
```

And $x = (1, 1, \cdots, 1)^T$ is the unique solution.

5.2. Computational Results

In order to validate the performance of quasi-Newton method and smooth Newton method [15], all problems with different dimensions are solved. Both two methods can better solve all AVE problems, and the only difference is computation time. Detailed time consumed (in seconds) by quasi-Newton method and smooth Newton method are shown in Table 1.

Table 1. Detailed Computational Results for Solving AVE

n	Problem 1		Problem 2		Problem 3	
	Literature [15]	Algorithm 4.1	Literature [15]	Algorithm 4.1	Literature [15]	Algorithm 4.1
8	2.2463e-01	1.9156e-01	2.0387e-01	2.2138e-01	2.2671e-01	4.6614e-01
16	1.9817e-01	1.7614e-01	2.0327e-01	3.3228e-01	2.0697e-01	3.1517e-01
32	1.3182e-01	2.9504e-01	3.1278e-01	4.0457e-01	4.2120e-01	4.5155e-01
64	2.6280e-01	3.8578e-01	7.5631e-01	9.2178e-01	8.2783e-01	8.0741e-01
128	5.7621e-01	6.0386e-01	1.8361e+00	1.7888e+00	2.0160e+00	1.9551e+00
256	2.3177e+00	1.8785e+00	7.1377e+00	6.8506e+00	7.7866e+00	6.5793e+00
512	1.5106e+01	1.1017e+01	3.8601e+01	3.2385e+01	4.4028e+01	3.2306e+01

From Table 1, with the dimension increased, our method is less time consumed in most cases, especially for larger dimension.

6. Conclusion

A new upper uniform smooth approximation function of absolute value function is presented, and is applied to solve AVE. The effectiveness of the method is demonstrated by solving some randomly generated AVE problems. Future works will also focus on studying the upper uniform smooth approximation function of absolute value function on other engineering problems, such as nonlinear control [17], support vector machines [18-19], artificial intelligence [20].

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