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Guaranteed Cost Control for Uncertain Neutral Systems with a Minimal Order Observer

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Abstract

This paper presents a design scheme of a minimal order observer-based guaranteed cost controller for uncertain neutral systems, in which some state variables can not be measured. The uncertainties are assumed to be norm-bounded. The initial state is assumed unknown but their mean and covariance are assumed known. A sufficient condition for robust stability analysis and robust stabilization are derived via linear matrix inequalities (LMIs). To show the advantage of the proposed method, a numerical example is given.

Keywords: guaranteed cost control, uncertain neutral systems, a minimal order observer, linear matrix inequalities (LMIs)

1. Introduction

Due to the presence of the uncertainties may cause instability, bad performance and lack of the exact model for the controlled systems, then considerable attention has been drawn to the problem of robust stability and stabilization for systems with parameter uncertainties [1]. Moreover, it has been devoted to find a controller which guarantees robust stability. Especially in a real plant control, it is also desirable to design a control system which not only achieves the stability but also guarantees an adequate level of performance [2].The guaranteed cost control is one approach to solve this kind of problem because it has an advantage in providing an upper bound on the quadratic cost function of the closed loop system. Lyapunov method via linear matrix inequality is often used to proof the stability criteria [3],[4]. Some research papers on guaranteed cost controller design can be found in [5],[6] and references therein.

In practical use, it is difficult to measure all system states that are needed in controller design because of some reasons such as poor plant knowledge, sensor avaibility, etc. On the other hand, the observer based control may be better than a state feedback one because it may not be possible to measure all system states. The observer could be embedded in the systems for either systems without all available states or systems with partially available states. A full order observer and a minimal order observer are applied to reconstruct the system states [7],[8]. However, a minimal order observer and reduced order observer are fewer investigated than a full order observer based guaranteed cost control research [9].

This paper will consider a minimal order observer based control to develop guaranteed cost control for uncertain neutral systems. The neutral system is system that depends on the delay of the state and its derivative. This type can be found in many fields of engineering and technology application such as in chemical process, networked control systems, robotic implementation etc [10]. The uncertainties are assumed to be norm-bounded with unknown initial state but their mean and covariance are known. Since the inverse relations are appeared in LMI solution, an iterative algorithm is used [11].

Notations. Throughout the paper, the superscripts ”*T*” and ”−*1*” stand for matrix transpose and inverse, $\Re^n$ denotes the n-dimensional Euclidean space, *X* >*Y* or *X* ≥ *Y* means that *X*−*Y* is positive definite or semi-positive definite, *I* is an identity matrix with appropriate dimensions, and * represents the symmetric elements in a symmetric matrix.

2. Problem Statements

Consider an uncertain neutral system with

$$\dot{x}(t) = A(t)x(t) + A_1(t)x(t - h(t)) + A_2(t)x(t - \tau) + B(t)u(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

$$x(t) = \varphi(t), \ t \in \{-h, 0\} \tag{3}$$

where $0 \leq h(t) \leq h$, $\dot{h}(t) \leq d$, h and τ are the given constant time delay in the states and their derivatives, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^r$ is the control input vector, $y(t) \in \mathbb{R}^m$ is the measured output vector, A, A_1, A_2, B, C are known constant real-valued matrices and C is restricted to the form of $C = [0 \quad I_m]$. The uncertainties in the states are written in

$$A(t) = A + D_A F_A(t)E_A, \ A_1(t) = A_1 + D_{A1} F_{A1}(t)E_{A1}, \ A_2(t) = A_2 + D_{A2} F_{A2}(t)E_{A2},$$
$$B(t) = B + D_B F_B(t)E_B \text{ which satisfy}$$
$$F_A^T(t)F_A(t) < I, \ F_{A1}^T(t)F_{A1}(t) < I, \ F_{A2}^T(t)F_{A2}(t) < I, \ F_B^T(t)F_B(t) < I \tag{4}$$

where $D_A, E_A, D_{A1}, E_{A1}, D_{A2}, E_{A2}, D_B, E_B$ are constant real-valued known matriceswith appropriate dimensions, and $F_A(t), F_{A1}(t), F_{A2}(t), F_B(t)$ are real time-varying unknown continuous and deterministic matrices.

We further assume that the initial state variable $x(0)$is unknown, but their mean and covariance are known, equivalently

$$E[x(0)] = m_0 \tag{5}$$

$$E[(x(0) - m_0)(x(0) - m_0)^T] = \Sigma_0 > 0 \tag{6}$$

$$z(0) - Tm_0 = 0 \tag{7}$$

where $E[\cdot]$ is the expected value operator.

The problem determined here is to design a minimal order observer

$$\dot{z}(t) = Dz(t) + Ey(t) + Fu(t) \tag{8}$$

$$\hat{x}(t) = Pz(t) + Wy(t) \tag{9}$$

and a controller

$$u(t) = K\hat{x}(t) \tag{10}$$

with $D = A_{11} + LA_{21}, \ PT + WC = I_n, \ F = TB, \ TA - DT = EC$,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{bmatrix}, \ P = [I_{n-m} \quad 0]^T, \ T = [I \quad L],$$

to achieve an upper bound of the following quadratic performance index

$$E[J] = E\left[\int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t) \, dt\right] \tag{11}$$

associated with the system (1)-(2) where Q, R are given symmetric positive definite matrices.

3. Main Result

In this section, a sufficient condition is established for the existence of a minimal order observer-based guaranteed cost controller for the uncertain system (1) and (2).

The gain of controller is formulated in:

$$K = -R^{-1}B^T S_1 \tag{12}$$

where S_1 is a symmetric positive definite matrix.

The main result of this study is given by Theorem 1.

Theorem 1. If the following matrix inequalities optimization problem;

$\min \{\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + tr(M_1) + tr(M_2) + tr(M_3) + tr(M_4)\}$ subject to

$$\begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 & \Lambda_4 & 0 & 0 & \Lambda_5 & 0 & \cdots \\ & \Lambda_6 & 0 & \Lambda_7 & 0 & 0 & \Lambda_8 & \Lambda_9 \\ & & \Lambda_{10} & \Lambda_{11} & 0 & 0 & \Lambda_{12} & \Lambda_{13} \\ & & & \Lambda_{14} & 0 & 0 & \Lambda_{15} & \Lambda_{16} \\ & & & & \Lambda_{17} & 0 & 0 & 0 & \vdots \\ & & & & & -U_2 & 0 & 0 \\ & & & & & & -U_1^{-1} & 0 \\ & & & & & & & -U_2^{-1} \\ & & & & & & & & \cdots \end{bmatrix}$$

$$\begin{bmatrix} \cdots & & \Delta_x & \\ & \ddots & & \\ & & diag\Delta_x & \Delta_\eta \\ & & & \ddots \\ & & & & diag\Delta_\eta \end{bmatrix} < 0 \tag{13}$$

$\sum_{k=1}^{p} e_{nk}^T \mathcal{O}_0 e_{nk} < \gamma_0, \ \sum_{k=1}^{m} e_{nk}^T \mathcal{O}_1 e_{nk} < \gamma_1, \ \sum_{k=1}^{m} e_{nk}^T \mathcal{O}_2 e_{nk} < \gamma_2, \ \sum_{k=1}^{m} e_{nk}^T \mathcal{O}_3 e_{nk} < \gamma_3 \tag{14}$

$$\begin{bmatrix} -\gamma_4 & v_1^T Y^T & v_2^T Y^T & \cdots & v_m^T Y^T \\ Yv_1 & -S_2 & & & \vdots \\ Yv_2 & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ Yv_m & \cdots & \cdots & \cdots & -S_2 \end{bmatrix} < 0 \tag{15}$$

$$\begin{bmatrix} -M_1 & E_x \\ * & -H_{1inv} \end{bmatrix} < 0 \tag{16}$$

$$\begin{bmatrix} -M_2 & F_x \\ * & -H_{2inv} \end{bmatrix} < 0 \tag{17}$$

$$\begin{bmatrix} -M_3 & G_x \\ * & -U_{1inv} \end{bmatrix} < 0 \tag{18}$$

$$\begin{bmatrix} -M_4 & H_x \\ * & -U_{2inv} \end{bmatrix} < 0 \tag{19}$$

Where

$$\Delta_1 = S_1(A + BK) + (A + BK)^T S_1 + H_1 + Q + K^T RK + (\alpha^{-1} + \alpha_1 + \beta_1 + \rho)E_A^T E_A,$$

$$\Delta_2 = S_1 A_1, \Delta_3 = S_1 A_2, \Delta_4 = S_1 BKP, \Delta_5 = (A + BK)^T U_1,$$

$$\Delta_6 = -(1 - d)H_1 + (\beta^{-1} + \mu + \alpha_3 + \beta_3)E_{A1}^T E_{A1}, \Delta_7 = -A_1^T T^T S_2,$$

$$\Delta_8 = A_1^T U_1, \Delta_9 = -A_1^T T^T U_2, \Delta_{10} = -U_1 + (e^{-1} + \omega + \alpha_4 + \beta_4)E_A^T E_A, \Delta_{11} = -A_2^T T^T S_2,$$

$$\Delta_{12} = A_2^T U_1, \Delta_{13} = -A_2^T T^T U_2, \Delta_{14} = S_2 D + D^T S_2 + H_2 + P^T K^T RKP,$$

$$\Delta_{15} = (BKP)^T U_1, \Delta_{16} = (BKP)^T T^T U_2, \Delta_{17} = -(1 - d)H_2,$$

$$\Delta_1 = S_1 D_A, \Delta_2 = S_1 D_{A1}, \Delta_3 = S_1 D_{A2}, \Delta_4 = S_1 D_B, \Delta_5 = K^T E_B^T,$$

$$\Delta_6 = U_1 D_A, \Delta_7 = U_1 D_B, \Delta_8 = U_1 D_{A1}, \Delta_9 = U_1 D_{A2},$$

$$\Delta_{10} = U_2 TD_A, \Delta_{11} = U_2 TD_B, \Delta_{12} = U_2 TD_{A1}, \Delta_{13} = U_2 TD_{A2}, \Delta_{14} = P^T K^T E_B^T,$$

$$\Delta_{15} = S_2 TD_A, \Delta_{16} = S_2 TD_B, \Delta_{17} = S_2 TD_{A1}, \Delta_{18} = S_2 TD_{A2},$$

$$\Delta_x = \{\Delta_1, \Delta_2, \cdots, \Delta_{13}\}, \Delta_\eta = \{\Delta_{14}, \Delta_{15}, \cdots, \Delta_{18}\},$$

$$diag \Delta_x = diag\{-\alpha^{-1}I, -\beta^{-1}I, -e^{-1}I, -\sigma^{-1}I, -(\alpha_1 + \alpha_2^{-1} + \beta_2^{-1})I, -(\alpha_2 + \alpha_2)I,$$

$$-\alpha_3 I, -\alpha_4 I, -\beta_1 I, -(\beta_2 + \beta_1)I, -\beta_3 I, -\beta_4 I\}$$

$$diag \Delta_\eta = diag\{-(\sigma + v + \alpha_2^{-1} + \beta_2)I, -\rho I, -(\lambda + v^{-1})I, -\mu I, -\omega I\},$$

has a set of solutions $S_1 > 0, S_2 > 0, H_1 > 0, H_2, U_1 > 0, U_2 > 0,$

satisfy the inverse relations $H_{1inv} = H_1^{-1}, H_{2inv} = H_2^{-1}, U_{1inv} = U_1^{-1}, U_{2inv} = U_2^{-1},$

$$\omega_{inv} = \sigma^{-1}, \lambda_{inv} = \lambda^{-1}, v_{inv} = v^{-1}, \alpha_{2inv} = \alpha_2^{-1}$$

then the minimal order observer law (8)-(10) is a guaranteed cost controller with a minimum expected value of guaranteed cost

$$E[J^*] = E\left[w^T(0)\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}w(0) + \int_{-h(t)}^0 w^T(t)\begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}w(t)\,dt + \int_{-\tau}^0 w^T(0)\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}w(0)\,dt\right] \quad (20)$$

where $w^T(\cdot) = \begin{bmatrix} x^T(\cdot) \\ \eta^T(\cdot) \end{bmatrix}$ and $\eta(t) = z(t) - Tx(t)$ is the estimated error of the minimal order observer.

Remark 1. Since (13)-(19) have constraints in inverse relations, an iterative LMI approach is applied to solve [11].

Before giving a proof of Theorem 1, a key lemma is introduced ([12]).
Lemma 1. Let D and E be matrices of appropriate dimensions, and F be a matrix function satisfying $F^T(t)F(t) \leq I$. Then for any positive scalar α, the following inequality holds

$$DFE + E^T F^T D^T \leq \alpha DD^T + \alpha^{-1}E^T E \quad (21)$$

Proof of Theorem 1
The closed loop system is obtained by mathematical substitution of (1)-(2) and (8)-(10)

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\eta}(t) \end{bmatrix} = \begin{bmatrix} A(t)+B(t)K & A_1(t) & A_2(t) & B(t)KP \\ -(T\Delta A+T\Delta BK) & -TA_1 & -TA_2 & D-T\Delta BKP \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t-\tau) \\ \eta(t) \end{bmatrix} \tag{22}$$

Define a candidate of Lyapunov function

$$V(t) = w^T(t)\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}w(t) + \int_{t-h(t)}^{t} w^T(t)\begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}w(t)\,dt + \int_{t-\tau}^{t} \dot{w}^T(t)\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}\dot{w}(t)\,dt \tag{23}$$

where $w^T(t) = \begin{bmatrix} x^T(t) \\ \eta^T(t) \end{bmatrix}$

Then, the time derivative of (23) along to (22) is calculated as

$$\dot{V}(t) = 2w^T(t)\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}\dot{w}(t) + w^T(t)\begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}w(t) - \left(1-\dot{h}(t)\right)w^T(t-h(t))\begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}$$

$$w(t-h(t)) + \dot{w}(t)\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}\dot{w}(t) - \dot{w}(t-\tau)\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}\dot{w}(t-\tau)$$

$$= \psi^T(t)\Omega(t)\psi(t) - \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) \tag{24}$$

where $\psi^T(t) = \begin{bmatrix} x^T(t) \\ x^T(t-h(t)) \\ \dot{x}^T(t-\tau) \\ \eta^T(t) \\ \eta^T(t-h(t)) \\ \eta^T(t-\tau) \end{bmatrix}$, $\Omega(t) = \begin{bmatrix} \Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 & 0 & 0 \\ & \Omega_5 & 0 & \Omega_6 & 0 & 0 \\ & & \Omega_7 & \Omega_8 & 0 & 0 \\ & & & \Omega_9 & 0 & 0 \\ & & & & -H_2(1-d) & 0 \\ & & & & & -U_2 \end{bmatrix}$,

$\Omega_1 = S_1(A(t)+B(t)K) + (A(t)+B(t)K)^T S_1 + (A(t)+B(t)K)^T U_1(A(t)+B(t)K)$
$- (T\Delta A(t)+T\Delta B(t)K)^T U_2(T\Delta A(t)+T\Delta B(t)K)$, $\Omega_2 = S_1 A_1(t)$, $\Omega_3 = S_1 A_2(t)$,

$\Omega_4 = S_1 B(t)KP - (T\Delta A(t)+T\Delta B(t)K)^T S_2$, $\Omega_5 = -H_1(1-d) + A_1^T(t)U_1 A_1(t) - A_1^T(t)T^T U_2 TA_1(t)$,

$\Omega_6 = -A_1^T(t)T^T S_2$, $\Omega_7 = -U_1 + A_2^T(t)U_1 A_2(t) - A_2^T(t)T^T U_2 TA_2(t)$, $\Omega_8 = -A_2^T(t)T^T S_2$,

$\Omega_9 = S_2(D-T\Delta B(t)KP) + (D-T\Delta B(t)KP)^T S_2 + H_2 + (B(t)KP)^T U_1 B(t)KP$
$+ (D-T\Delta B(t)KP)^T U_2(D-T\Delta B(t)KP)$,

Under condition

$$\Omega(t) < 0 \tag{25}$$

equation (24) leads to

$$\dot{V}(t) < -\left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) < 0 \tag{26}$$

for any $x(t) \neq 0$ and the closed loop system is asymptotically stable.

Here, condition (24) is investigated by applying *lemma 1* as follows

$$2x^T(t)S_1\Delta A(t)x(t) = 2x^T(t)S_1D_AF_A(t)E_Ax(t) \le \alpha x^T(t)S_1D_AD_A^TS_1x(t) + \alpha^{-1}x^T(t)E_A^TE_Ax(t) \quad (27)$$

$$\begin{aligned}
2x^T(t)S_1\Delta A_1(t)x(t-h(t)) &= 2x^T(t)S_1D_{A1}F_{A1}(t)E_{A1}x(t-h(t))\\
&\le \beta x^T(t)S_1D_{A1}D_{A1}^TS_1x(t) + \beta^{-1}x^T(t-h(t))E_{A1}^TE_{A1}x(t-h(t))
\end{aligned} \quad (28)$$

$$\begin{aligned}
2x^T(t)S_1\Delta A_2(t)x(t-\tau) &= 2x^T(t)S_1D_{A2}F_{A2}(t)E_{A2}x(t-\tau)\\
&\le \epsilon x^T(t)S_1D_{A2}D_{A2}^TS_1x(t) + \epsilon^{-1}x^T(t-\tau)E_{A2}^TE_{A2}x(t-\tau)
\end{aligned} \quad (29)$$

$$\begin{aligned}
2x^T(t)S_1\Delta B(t)KP\eta(t) &= 2x^T(t)S_1D_BF_B(t)E_BKP\eta(t)\\
&\le \sigma x^T(t)S_1D_BD_B^TS_1x(t) + \sigma^{-1}\eta^T(t)P^TK^TE_B^TE_BKP\eta(t)
\end{aligned} \quad (30)$$

$$\begin{aligned}
-2x^T(t)\Delta A^T(t)T^TS_2\eta(t) &= -2x^T(t)E_A^TF_A^T(t)D_A^TT^TS_2\eta(t)\\
&\le \rho x^T(t)E_A^TE_Ax(t) + \rho^{-1}\eta^T(t)S_2TD_AD_A^TT^TS_2\eta(t)
\end{aligned} \quad (31)$$

$$\begin{aligned}
-2x^T(t)K^T\Delta B^T(t)T^TS_2\eta(t) &= -2x^T(t)K^TE_B^TF_B^T(t)D_B^TT^TS_2\eta(t)\\
&\le \lambda x^T(t)K^TE_B^TE_BKx(t) + \lambda^{-1}\eta^T(t)S_2TD_BD_B^TT^TS_2\eta(t)
\end{aligned} \quad (32)$$

$$\begin{aligned}
-2x^T(t-h(t))\Delta A_1^T(t)T^TS_2\eta(t) &= -2x^T(t-h(t))E_{A1}^TF_{A1}^T(t)D_{A1}^TT^TS_2\eta(t)\\
&\le \mu x^T(t-h(t))E_{A1}^TE_{A1}x(t-h(t)) + \mu^{-1}\eta^T(t)S_2TD_{A1}D_{A1}^TT^TS_2\eta(t)
\end{aligned} \quad (33)$$

$$\begin{aligned}
-2x^T(t-\tau)\Delta A_2^T(t)T^TS_2\eta(t) &= -2x^T(t-\tau)E_{A2}^TF_{A2}^T(t)D_{A2}^TT^TS_2\eta(t)\\
&\le \omega x^T(t-\tau)E_{A2}^TE_{A2}x(t-\tau) + \omega^{-1}\eta^T(t)S_2TD_{A2}D_{A2}^TT^TS_2\eta(t)
\end{aligned} \quad (34)$$

$$\begin{aligned}
-2\eta^T(t)S_2T\Delta B(t)KP\eta(t) &= -2\eta^T(t)S_2TD_BF_B(t)E_BKP\eta(t)\\
&\le v\eta^T(t)S_2TD_BD_B^TT^TS_2\eta(t) + v^{-1}\eta^T(t)P^TK^TE_B^TE_B\eta(t)
\end{aligned} \quad (35)$$

$$\begin{aligned}
2x^T(t)\Delta A^T(t)U_1x(t) &= 2x^T(t)E_A^TF_A^T(t)D_A^TU_1x(t)\\
&\le \alpha_1 x^T(t)E_A^TE_Ax(t) + \alpha_1^{-1}x^T(t)U_1D_AD_A^TU_1x(t)
\end{aligned} \quad (36)$$

$$\begin{aligned}
2x^T(t)K^T\Delta B^T(t)U_1x(t) &= 2x^T(t)K^TE_B^TF_B^T(t)D_B^TU_1x(t)\\
&\le \alpha_2 x^T(t)K^TE_B^TE_BKx(t) + \alpha_2^{-1}x^T(t)U_1D_BD_B^TU_1x(t)
\end{aligned} \quad (37)$$

$$\begin{aligned}
2x^T(t-h(t))\Delta A_1^T(t)U_1x(t) &= 2x^T(t-h(t))E_{A1}^TF_{A1}^T(t)D_{A1}^TU_1x(t)\\
&\le \alpha_3 x^T(t-h(t))E_{A1}^TE_{A1}x(t-h(t)) + \alpha_3^{-1}x^T(t)U_1D_{A1}D_{A1}^TU_1x(t)
\end{aligned} \quad (38)$$

$$\begin{aligned}
2x^T(t-\tau)\Delta A_2^T(t)U_1x(t) &= 2x^T(t-\tau)E_{A2}^TF_{A2}^T(t)D_{A2}^TU_1x(t)\\
&\le \alpha_4 x^T(t-\tau)E_{A2}^TE_{A2}x(t-\tau) + \alpha_4^{-1}x^T(t)U_1D_{A2}D_{A2}^TU_1x(t)
\end{aligned}$$

$$\begin{aligned}
2\eta^T(t)P^TK^T\Delta B^T(t)U_1\eta(t) &= 2\eta^T(t)P^TK^TE_B^TF_B^T(t)D_B^TU_1\eta(t)\\
&\le \alpha_5\eta^T(t)P^TK^TE_B^TE_BKP\eta(t) + \alpha_5^{-1}x^T(t)U_1D_BD_B^TU_1x(t)
\end{aligned} \quad (39)$$

$$\begin{aligned}
-2x^T(t)\Delta A^T(t)T^TU_2x(t) &= -2x^T(t)E_A^TF_A^T(t)D_A^TT^TU_2x(t)\\
&\le \beta_1 x^T(t)E_A^TE_Ax(t) + \beta_1^{-1}x^T(t)U_2TD_AD_A^TT^TU_2x(t)
\end{aligned} \quad (40)$$

$$\begin{aligned}
-2x^T(t)K^T\Delta B^T(t)T^TU_2x(t) &= -2x^T(t)K^TE_B^TF_B^T(t)D_B^TT^TU_2x(t)\\
&\le \beta_2 x^T(t)K^TE_B^TE_BKx(t) + \beta_2^{-1}x^T(t)U_2TD_BD_B^TT^TU_2x(t)
\end{aligned} \quad (41)$$

$$\begin{aligned}
-2x^T(t-h(t))\Delta A_1^T(t)T^TU_2x(t) &= -2x^T(t-h(t))E_{A1}^TF_{A1}^TD_{A1}^TT^TU_2x(t)\\
&\le \beta_3 x^T(t-h(t))E_{A1}^TE_{A1}x(t-h(t)) + \beta_3^{-1}x^T(t)U_2TD_{A1}D_{A1}^TT^TU_2x(t)
\end{aligned} \quad (42)$$

$$\begin{aligned}
-2x^T(t-\tau)\Delta A_2^T(t)T^TU_2x(t) &= -2x^T(t-\tau)E_{A2}^TF_{A2}^TD_{A2}^TT^TU_2x(t)\\
&\le \beta_4 x^T(t-\tau)E_{A2}^TE_{A2}x(t-\tau) + \beta_4^{-1}x^T(t)U_2TD_{A2}D_{A2}^TT^TU_2x(t)
\end{aligned} \quad (43)$$

$$-2\eta^T(t)P^TK^T\Delta B^T(t)T^TU_2\eta(t) = -2\eta^T(t)P^TK^TB_B^TF_B^T(t)D_B^TT^TU_2\eta(t)$$
$$\leq \beta_2\eta^T(t)P^TK^TB_B^TB_BKP\eta(t) + \beta_2^{-1}x^T(t)U_2TD_BD_B^TT^TU_2x(t) \tag{44}$$

Applying lemma 1., denoting $H_{1inv} = H_1^{-1}$, $H_{2inv} = H_2^{-1}$, $U_{1inv} = U_1^{-1}$, $U_{2inv} = U_2^{-1}$, $\sigma_{inv} = \sigma^{-1}$, $\lambda_{inv} = \lambda^{-1}$, $v_{inv} = v^{-1}$, $\alpha_{2inv} = \alpha_2^{-1}$, and using schur complement lead to (13)

Further, integrating (26) from 0 to T infinity, we have

$$J = \int_0^\infty (x^T(t)Qx(t) + u^T(t)Ru(t)) \, dt$$
$$< x^T(0)S_1x(0) + \eta^T(0)S_2\eta(t) + \int_{-h(v)}^0 x^T(s)H_1x(s)ds + \int_{-h(v)}^0 \eta^T(s)H_2\eta(s)ds +$$
$$\int_{-\tau}^0 x^T(s)U_1x(s)ds + \int_{-\tau}^0 \eta^T(s)U_2\eta(s)ds = J^* \tag{45}$$

Consider the optimal expected value of the guaranteed cost, we have

$$E[J^*] = tr \, S_1 \, E[x(0)x^T(0)] + tr \, S_2 \, E[\eta(0)\eta^T(0)] + trH_1E\left[\int_{-h(v)}^0 x(s)x^T(s)ds\right]$$
$$trH_2E\left[\int_{-h(v)}^0 \eta(s)\eta^T(s)ds\right] + trU_1E\left[\int_{-\tau}^0 x(s)x^T(s)ds\right] + trU_2E\left[\int_{-\tau}^0 \eta(s)\eta^T(s)ds\right] \tag{46}$$

A relation between mean $m(t)$ and covariance $\Sigma(t)$ of the expected value is

$$\Sigma(t) = E[x(t)x^T(t)] - m(t)m^T(t) \tag{47}$$

Substituting (47) into (46), we obtain

$$E[J^*] = tr \, S_1 \, E[\Sigma_0 + m_0m_0^T] + tr \, S_2 \, E[(z(0) - Tx(0))(z(0) - Tx(0))^T]$$
$$+trH_1E\left[\int_{-h(v)}^0 (\Sigma(s) + m(s)m^T(s))ds\right] + trH_2E\left[\int_{-h(v)}^0 \eta(s)\eta^T(s)ds\right]$$
$$+trU_1E\left[\int_{-\tau}^0 x(s)x^T(s)ds\right] + trU_2E\left[\int_{-\tau}^0 \eta(s)\eta^T(s)ds\right] \tag{48}$$

It is easily seen that

$$E[(z(0) - Tx(0))(z(0) - Tx(0))^T] = T\Sigma_0T^T + (z(0) - Tx(0))(z(0) - Tx(0))^T \tag{49}$$

$$trH_1E\left[\int_{-h(v)}^0 x(s)x^T(s)ds\right] = tr \, H_1E_XE_X^T \tag{50}$$

$$trH_2E\left[\int_{-h(v)}^0 \eta(s)\eta^T(s)ds\right] = tr \, H_2F_XF_X^T \tag{51}$$

$$trU_1E\left[\int_{-\tau}^0 x(s)x^T(s)ds\right] = tr \, U_1G_XG_X^T \tag{52}$$

$$trU_2E\left[\int_{-\tau}^0 \eta(s)\eta^T(s)ds\right] = tr \, U_2H_XH_X^T \tag{53}$$

Here, we consider positive scalars

$$tr \, S_1 \, E[\Sigma_0 + m_0m_0^T] < \gamma_0, \tag{54}$$

$$tr \, S_2\Sigma_{11} < \gamma_1, \tag{55}$$

$$tr \, S_2L\Sigma_{21} < \gamma_2 \tag{56}$$

$$tr \, S_2\Sigma_{12}L^T < \gamma_3 \tag{57}$$

$$tr \, S_2L\Sigma_{22}L^T < \gamma_4 \tag{58}$$

$$tr\ H_1 E_X E_X^T \prec tr\ M_1 \tag{59}$$

$$tr\ H_2 f_X F_X^T \prec tr\ M_2 \tag{60}$$

$$tr\ U_1 G_X G_X^T \prec tr\ M_3 \tag{61}$$

$$tr\ U_2 H_X H_X^T \prec tr\ M_4 \tag{62}$$

Minimizing $\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + tr\ M_1 + tr\ M_2 + tr\ M_3 + tr M_4$ results in min $E[J^*]$. By recalling $tr(AB) = tr(BA)$, (54)-(57) lead to (14) and (59)-(62) to (16)-(19).

By denoting $\Sigma_{22}^{\frac{1}{2}} = [v_1, v_2, \cdots, v_m]$, (58) is calculated as

$$tr\ S_2 L \Sigma_{22} L^T = v_1^T Y^T S_2^{-1} Y v_1 + v_2^T Y^T S_2^{-1} Y v_2 + \cdots + v_m^T Y^T S_2^{-1} Y v_m$$

$$= [v_1^T Y^T \quad v_2^T Y^T \quad \cdots \quad v_m^T Y^T] S_2^{-1} \begin{bmatrix} Y v_1 \\ Y v_2 \\ \vdots \\ Y v_m \end{bmatrix} \prec \gamma_4 \tag{63}$$

Schur complement derives (15) from (63)

4. Numerical example

Consider system (1)-(2) with

$$A = \begin{bmatrix} -3 & 0 \\ -2 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & -2 \\ 0 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, C = [0 \quad 1], m_0 = 0_2, \Sigma_0 = I_2,$$

$$R = 2, Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, h = 0.5, \tau = 0.4, d = 0.2$$

$$D_A = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, E_A = \begin{bmatrix} 0.3 & 0.3 \\ 0 & 0.3 \end{bmatrix}, D_{A1} = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, E_{A1} = \begin{bmatrix} 0.3 & -0.3 \\ 0 & -0.3 \end{bmatrix},$$

$$D_{A2} = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, E_{A2} = \begin{bmatrix} 0.3 & -0.3 \\ 0 & 0.3 \end{bmatrix}, D_B = \begin{bmatrix} 0.3 & 0 \\ 0 & 0 \end{bmatrix}, E_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

Applying the minimal-order observer-based approach [8],[9], we obtain the solutions of controller and observer gain:

$$K = [0.0308 \quad -0.4064], L = -0.1132,$$

guaranteed cost value $J^* = 7.9697$ and other set of solutions as follows

$$S_1 = \begin{bmatrix} 0.1741 & -0.0002 \\ -0.0002 & -0.1768 \end{bmatrix}, S_2 = 0.2928, H_1 = \begin{bmatrix} 1.1125 & -0.0843 \\ -0.0843 & 1.1968 \end{bmatrix}, H_2 = 0.9188$$

$$U_1 = \begin{bmatrix} 0.9748 & 0.0995 \\ 0.0995 & 1.1345 \end{bmatrix}, U_2 = 0.2321, \alpha = 0.3793, \tau = 0.9483, Y = -0.0331,$$

$$Z = [0.2928 \quad -0.0331].$$

Figure 1 shows the trajectory of states with initial condition for nominal and neutral systems $x(0) = [1 \quad -1]^T$, $x_h(0) = [0 \quad 0]^T$, $x_n(0) = [0 \quad 0]^T$. It is shown that the states converge to the stable condition. Hence the controller and observer gain are designed well for the uncertain neutral systems with a minimal order observer.
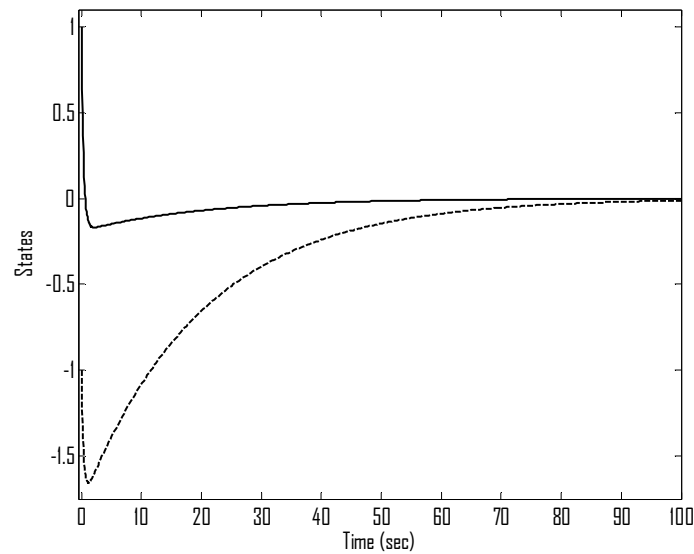
Figure 1. Trajectory of states x_1 (–) and x_2 (- -)

5. Conclusion

This paper discusses a guaranteed cost controller design for uncertain neutral systems with a minimal order observer. The stability on the LMIs forms is determined by deriving a sufficient condition for the existence of state feedback controller. To illustrate the proposed method, a numerical simulated example is presented. The problem of the guaranteed cost controller for uncertain systems is still open according to the advantage of this method. In the future work, we will investigate the problem of discrete systems.

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