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Asymptotic features of Hessian Matrix in Receding Horizon Model Predictive Control with Medium Sized Prediction Frames

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Abstract

In this paper, Receding Horizon Model Predictive Control (RH-MPC) having a quadratic objective function is studied through the Singular Value Decomposition (SVD) and Singular Vectors of its Hessian Matrix. Contrary to the previous work, non-equal and medium sized control and prediction horizons are considered and it is shown that the Singular Values converge to the open loop magnitude response of the system and singular vectors contain the phase information. Earlier results focused on classical formulation of Generalized Predictive Control (GPC), whereas, current work proves the applicability to modern formulation. Although, method can easily be extended to MIMO systems, only SISO system examples are presented.

Keywords*: MPC, receding horizon MPC, frequency response, hessian matrix, singular value decomposition*

1. Introduction

Model Predictive Control is a time domain class of control strategies, with very attractive features suitable for modern digital applicability. Although, the idea of the model based predictive controller is classic, it has become recently a standardized technique of choice for slow varying processes especially found in the chemical and petrochemical industries. The area is currently under intensive research only to extend its usefulness to other fields and applications such as fast varying processes, control of power converters and hybrid control systems.

Ability to deal with Multi-Input Multi-Output (MIMO) systems of higher orders, effectively dealing with constraints imposed on input, change in input, output and state variables are the unique features of MPC that make it ideal. The basic idea behind Receding Horizon MPC (RH-MPC) is to solve online, on each sampling interval, a standard optimization open loop problem to calculate optimal control law. Only the first element of the resulting control signal is actually applied and the problem is solved again on the next sampling interval. Plant model with incorporated constraints is predefined in the controller and control law is determined based on this model, hence any discrepancies in the model result in unwanted consequences [1].

The performance of predictive controller deteriorates over time if proper care of the plant is not taken. A slight change in the parameters of the plant creates model-mismatch and accuracy of the calculated law is disturbed which must be robust in nature. It is well known that uncertainty is better understood in frequency domain representation of the model. However, no solid frequency domain connection of the RH-MPC exists which can provide an insight into the well-established frequency characteristics of the problem. As mentioned in [1-6] MPC optimization must be restricted to operate near the frequencies where plant model is accurate. It also affects the stability of the overall performance and bandwidth of operation.

Some work has been done into the area by Rojas and Goodwin [6-7] and Quang et el [6] to formulate the connection between the open loop frequency response and Singular Value Decomposition (SVD) of the Toeplitz matrix of MPC. In [7] it has been shown that singular values of the Hessian matrix are related to the frequency domain gain of related system. Similar results are also reported by [6] where open loop phase response is shown to be

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associated with singular vectors of the Toeplitz matrix for both SISO and MIMO systems. Other such studies include Ludlage [6], Sanchis et al. [9] etc. In [8] and [6], the studies are carried out under sufficiently long prediction horizon and restricted choices for the weighting matrices of the quadratic objective function. Control horizons are assumed to be equal to prediction horizon which is practically infeasible. In [6] frequency response connection is established to the classical generalized predictive control (GPC), whereas, we validate the results for receding horizon MPC and non-equal prediction and control horizons. It is suggested that curve fitting techniques can be utilized to form the proper magnitude and phase response for smaller prediction horizons. Earlier studies assumed that no weight is imposed on inputs; however, we show that for non-zero input penalty, Cholesky decomposition of the associated Hessian can lead to investigation of the SVD of Toeplitz matrix [10-11]. The paper gives an overview of RH-MPC formulation, Hessian matrix and Toeplitz matrix in section 2. In section 3 SVD of Toeplitz is considered in detail and connection are presented and established to the frequency response of the associated open loop system. Some examples and results are given at the end in section IV.

2. Recending Horizon Model Predictive Control(RH-MPC)

Consider the discrete state space representation of a SISO plant with no effect of input on output i.e. [7-8].

$$D_p = 0$$

$$x_p[n+1] = A_p x_p[n] + B_p u[k] \tag{1}$$

$$y[n] = C_p x_p[n] \tag{2}$$

Where u the plant is is input; y is the output and x_p defines the state vector of dimension p. In receding horizon MPC, the next control signal is defined as the change in previous input and not the altogether new control signal. Taking the difference in (1) and (2) leads to

$$\Delta x_p[n+1] = A_p \Delta x_p[n] + B_p \Delta u[n] \tag{3}$$

$$\begin{aligned} y[n+1] - y[n] &= C_p \Delta x_p[n+1] \\ y[n+1] &= C_p A_p \Delta x_p[n] + C_p B_p \Delta u[n] + y[n] \end{aligned} \tag{4}$$

The differences are defined as:

$$\begin{aligned} \Delta x_p[n+1] &= x_p[n+1] - x_p[n], \\ \Delta x_p[n] &= x_p[n] - x_p[n-1], \\ \Delta u[n] &= u[n] - u[n-1] \end{aligned}$$

Combining (3) and (4) gives the augmented state space representation suitable for RH-MPC [14]:

$$\underbrace{\begin{bmatrix} \Delta x_p[n+1] \\ y[n+1] \end{bmatrix}}_{x[n+1]} = \underbrace{\begin{bmatrix} A_p & 0_{p\times 1} \\ C_p A_p & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \Delta x_p[n] \\ y[n] \end{bmatrix}}_{x[n]} + \underbrace{\begin{bmatrix} B_p \\ C_p B_p \end{bmatrix}}_{B} \Delta u[n]$$

$$y[n] = \underbrace{\begin{bmatrix} 0_{1\times p} & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \Delta x_p[n] \\ y[n] \end{bmatrix}}_{x[n]}$$

Let N_c and N_p denote the control and prediction horizons, then using (5), we can write down the predicted outputs and states as follows:

$$Y = Ox[n] + T\Delta U \tag{5}$$

Where

$$Y = \left[y[n+1], y[n+2], ... y[n+N_p] \right]^T$$
$$O = \left[CA \quad CA^2 \quad CA^3 \quad \dots \quad CA^{N_p} \right]^T$$
$$\Delta U = \left[\Delta u[n] \quad \Delta u[n+1] \quad ... \quad \Delta u[n+N_c-1] \right]^T$$

And the Toeplitz matrix is defined as:

$$T = \begin{bmatrix} CB & 0 & 0 & \dots & 0 \\ CAB & CB & 0 & \dots & 0 \\ CA^2B & CAB & CB & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ CA^{N_p-1}B & CA^{N_p-2}B & CA^{N_p-3}B & \dots & CA^{N_p-N_c}B \end{bmatrix} \tag{6}$$

The control objective is reflected by the following cost function

$$J = (S-Y)^T Q (S-Y) + \Delta U^T R \Delta U \tag{7}$$

Here S defines the set point vector and predicted outputs Y are subtracted to minimize the error. Q and R are input and output weighting matrices. To minimize the quadratic cost function defined in (8), we need to set its derivative, with respect to the control input, equal to zero and determine the optimal control law.

Substituting (6) into (8) and setting $\frac{\partial J}{\partial \Delta U} = 0$ leads to the solution

$$\Delta U_{opt} = \left(T^T T + R\right)^{-1} T^T \left(S - Ox[n]\right) \tag{8}$$

It is assumed here that $\left(T^T T + R\right)^{-1}$ exists. The matrix $\left(T^T T + R\right)^{-1}$ is called Hessian and plays very important role in optimization of the objective function. In receding horizon model predictive control, ΔU_{opt} is determined at each sampling time instant. It contains N_c elements but only first element is actually applied as the controlling input to minimize the error and the whole prediction is made again at the next sampling interval. If $N_p > N_c$ (which is usually the case), then elements of ΔU_{opt} beyond N_c are assumed zero i.e. no change in the control signal [6], [15-16]. If the underlying system is assumed to be linear time-invariant (LTI), then its transfer function can be formed by taking the z-transform of (5), which leads to

$$Y[z] = \left(C(zI-A)^{-1}B\right)\Delta U[z]$$
$$H[z] = \frac{Y[z]}{\Delta U[z]} = \left(C(zI-A)^{-1}B\right) \tag{9}$$

The output of LTI system is also defined by the convolution sum relating impulse response and input vector

$$Y[n] = \sum_{k=0}^{\infty} h(k)\Delta U(n-k) \qquad (10)$$

Close inspection of (10) and (11), reveals that impulse response $h[n]$ is directly related to the first column of Toeplitz matrix defined in (7). (Note that inverse z-transform of (10) produces terms like CA^kB) [11]. Therefore, Toeplitz matrix can be described in terms of impulse response of the underlying system as:

$$T = \begin{bmatrix} h[0] & 0 & \cdots & 0 \\ h[1]+h[0] & h[0] & \cdots & 0 \\ h[2]+h[1] & h[1]+h[0] & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ h[N_p-1]+h[N_p-2] & h[N_p-2]+h[N_p-3] & \cdots & h[N_p-N_c]+h[N_p-N_c-1] \end{bmatrix} \qquad (11)$$

The special structure of the above matrix is due to the fact that state space formulation is based on augmented state space of the original system, hence resulting in the augmented Toeplitz matrix. The matrix can easily be converted to simple impulse response column vector by taking the difference of two consecutive elements in the first column. If we assume no weighting on inputs ($R=0$), then Hessian simplifies to $H=\left(T^TT\right)^{-1}$ provided that the inverse exists.

3. Frequency Domain Features Of The Hessian Matrix

In [6-7] and [1], various frequency domain characteristics of the Toeplitz matrix of the generalized predictive control are studied through singular value decomposition (SVD). The study differs in the special formulation of the Hessian in (12) while assuming $Q=I, R=0$ and equal sufficiently long prediction and control horizons. It is shown that singular values converge to the magnitude response of the system for a specific frequency, hence predicting the frequency domain characteristics from the time domain formulation of the MPC (Toeplitz matrix). Shows the connection between left and right singular vectors of Hessian and phase response of the open loop system under consideration [11]. In short, all the frequency domain analysis features (stability, open loop gain, and magnitude and phase response) can be extracted from the SVD of the Hessian [19]. We extended the work for Receding Horizon MPC formulation (augmented state space formulation) and studied it for different cases, and propose methods to seek frequency domain information for medium sized horizons. Approximating the studied system as finite impulse response (FIR) system, we can assume that impulse response of the discrete system becomes insignificant after a certain time i.e. $h[n]=0$ for $n=k_0 \quad and \quad k_0<N_p,N_c$. Normally we would like to assume $k_0 = (N_c-1)/2$. This assumption leads to following modified form of the Toeplitz matrix in (12) [17-18] and [19]:

$$T = \begin{bmatrix} h[0] & 0 & \cdots & & & 0 \\ \vdots & \ddots & & & & \vdots \\ h[k_0]+h[k_0-1] & \ldots & h[0] & 0 & & \vdots \\ \vdots & \ddots & & & \ddots & 0 \\ 0 & \cdots & & h[k_0]+h[k_0-1] & \cdots & h[N_c]+h[N_c-1] \end{bmatrix} \qquad (12)$$

Where $T \in C^{N_p \times N_c}$. The frequency response of the underlying open loop stable system is denoted by $G\left(e^{j\omega}\right)$ and if we decompose this system into its SVD, then we can write:

$$G\left(e^{j\omega}\right)=U\left(e^{j\omega}\right)S(\omega)V\left(e^{j\omega}\right) \tag{13}$$

Similarly the SVD decomposition of non-square matrix T produces three matrices $U \in C^{N_p \times N_p}$, $S \in C^{N_p \times N_c}$ and $V \in C^{N_c \times N_c}$. In [Rojas] it is proved that for a certain prediction horizon there exists a singular value of the Hessian matrix that is very close to density spectrum ($d\left(\omega_0\right)$) at some particular frequency ω_0. Mathematically it can be described as [20]:

$$H\upsilon_m^{N_p} \rightarrow d\left(\omega_0\right)\upsilon_m^{N_p} \quad N_p \rightarrow \infty \tag{14}$$

Where H is the Hessian matrix and $\upsilon_m^{N_p}$ is defines the following unit Euclidian norm vector i.e. $\left\|\upsilon_m^{N_p}\right\|_2 = 1$

$$\upsilon_m^{N_p} = \frac{1}{\sqrt{N_p}}\begin{bmatrix} 1 & e^{-j\frac{2\pi}{Np}m} & e^{-j\frac{2\pi}{Np}2m} & \cdots & e^{-j\frac{2\pi}{Np}(N_c-1)m} \end{bmatrix} \tag{15}$$

Here $m \in \mathbf{Z} \quad and \quad 0 \le m \le N_c - 1$ (a total of N_c elements for $N_c \neq N_p$). It is also shown in [Tran] that (15) also holds true for a non-exponential, sinusoidal elements vector defined as follows:

$$\upsilon_m^{N_p} = \frac{1}{L}\left[\cos\left(\phi_m\right) \quad \cos\left(-\frac{\pi}{N_p}m+\phi_m\right) \quad \cdots \quad \cdots \quad \cos\left(-\frac{\pi}{N_p}\left(N_c-1\right)m+\phi_m\right)\right]^T \tag{16}$$

$\phi_m \in R$ and L is chosen such that $\left\|\upsilon_m^{N_p}\right\|_2 = 1$. The auto-correlation sequence of the impulse response of the system can be shown connected with Hessian as explained through the following mathematical relations. We can define auto-correlation sequence as [21]

$$\lambda_m = \sum_{k=0}^{k_0-m} h[k]h[k+m] \quad for\ 0 \le m \le k_0 \tag{17}$$

Investigating the special Hessian structure formed by $H = T^T T$ for T defined in (13), one can easily establish the connection between auto-correlation (18) and Hessian. We also know that energy density spectrum is the Discrete Time Fourier Transform (DTFT) of (18) i.e.

$$d\left(\omega\right) = \sum_{m=0}^{\infty} \lambda_m e^{-j\omega m}, \quad \omega = \left[-\pi, \pi\right] \tag{18}$$

The above relation gives indirect information about frequency response of the system based on Hessian matrix keeping in mind that $\lambda_{-m} = \lambda_m$. For a detailed discussion of the topic see [13-14] and [11] for example. The singular value decomposition of toeplitz matrix reveals interesting facts about frequency response of the underlying system. We can state this information in a modified form as compared to the form given in [Tran] as follows: "Given a SISO discrete system $G(z)$, we can form its toplitz matrix T and its SVD: $T = USV^T$ where $U = \begin{bmatrix} U_1 & U_1 & \cdots & U_{N_p} \end{bmatrix}$ and $V = \begin{bmatrix} V_1 & V_1 & \cdots & V_{N_c} \end{bmatrix}$ are left and right singular vectors then frequency response can be extracted as given in following relations". [22]

$$\left|G\left(e^{j\omega}\right)\right| \rightarrow diag\left(S\right) \tag{19}$$

$$\angle G\left(e^{j\omega}\right) = \lim_{N_c \rightarrow \infty} \cos^{-1}\left(U_n^T V_n\right) \quad for \quad n = 0,1,...,N_c \tag{20}$$

V_n and U_n have the form given in (17).

4. Results And Examples

To illustrate the applicability of the above formulation, let us consider a second order system:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & -0.2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0.8 \end{bmatrix} x + \begin{bmatrix} 1 \end{bmatrix} u \end{aligned} \tag{21}$$

Setting sampling time $Ts = 1$ and discretizing the continuous system results in the following discrete version:

$$\begin{aligned} x[\mathrm{k}+1] &= \begin{bmatrix} 0.316 & -0.1223 \\ 0.6116 & 0.9276 \end{bmatrix} x[\mathrm{k}] + \begin{bmatrix} 0.6116 \\ 0.3622 \end{bmatrix} u[\mathrm{k}] \\ y[\mathrm{k}] &= \begin{bmatrix} 1 & 0.8 \end{bmatrix} x[\mathrm{k}] + \begin{bmatrix} 1 \end{bmatrix} u[\mathrm{k}] \end{aligned} \tag{22}$$

We calculated the Model Predictive Controller gains and related matrices for various values of control and prediction horizons and under different scenarios. For example choosing $k_o = 3$ and $N_c = N_p = 7$ results in toeplitz and Hessian of order 7×7 and it can be obtained by utilizing MPC algorithm described above. Figure 1 shows the frequency response of the system along with SVD corresponding features i.e. singular values plotted with magnitude response and singular vectors with phase response

Increasing the value of prediction horizon will improve the closeness of the related characteristics. Figure 2 shows the response for $k_o = 15$, $N_c = N_p = 31$. It is obvious from these results that accuracy becomes better with increasing the horizons. Similarly when we set different values of control and prediction horizon, the results still validate the relationship between frequency response and Hessian SVD of the system. Figure 3 gives the outcome when we set $N_p = 31$ and $N_c = 16$. The DC gain difference is removed by increasing horizons. However, it can also be ignored since it can also be calculated from the time domain information.

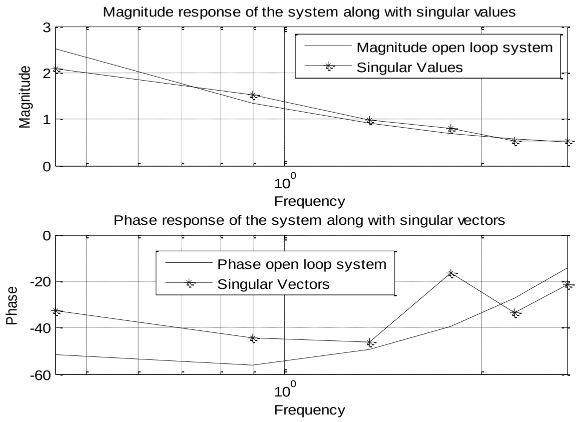

Figure 1. Frequency response approximation from SVD using Np=Nc=7

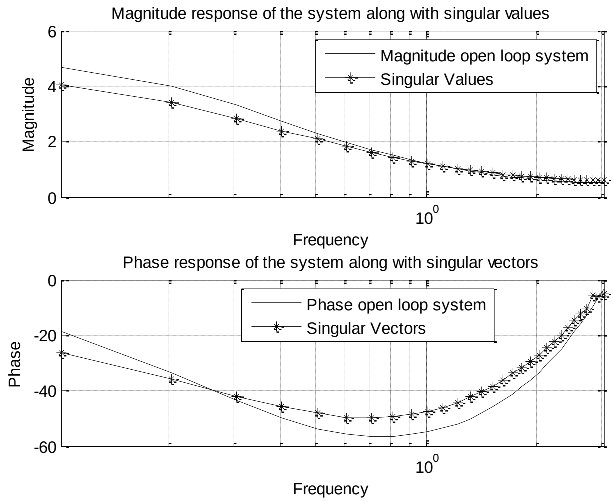

Figure 2. Frequency response approximation from SVD using Np=Nc=31

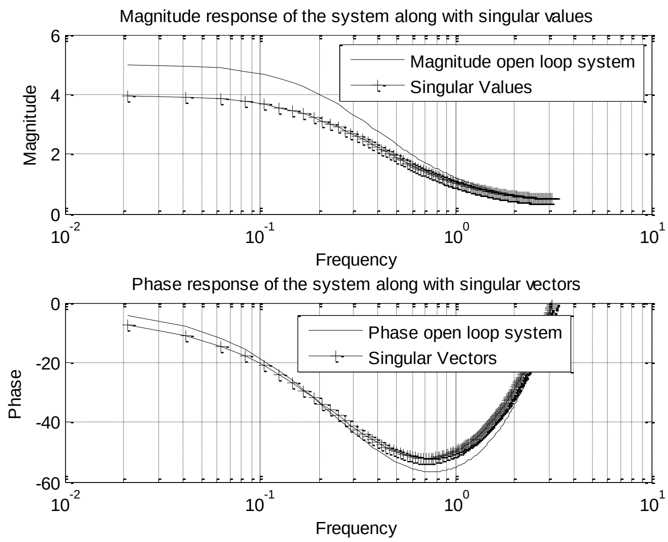

Figure 3. Frequency response approximation from SVD using Np=31, Nc=16

It is evident from the results above that as we increase the prediction horizon, frequency response is better approximated by the singular value information of the Hessian matrix. Ideally, both response are equal for infinite horizon. However, reasonable approximations are achieved even with finite short horizons to provide insight into the frequency domain characteristics of the systems using MPC receding horizon controllers.

5. Conclusion

The frequency response of a receding horizon model predictive control strategy is approximated by the singular value decomposition for medium horizons. It is shown that singular values of the Hessian converge to the open loop magnitude response of the system while singular vectors provide the phase insights. The approximations are also shown to hold valid for non-equal prediction and control horizons. Further work can be done to relate SVD to the closed loop frequency response of the Model predictive control and stability of the systems. For, fast dynamics, the horizons used here are still impractical, hence leaving the space for exploration of the subject for shorter horizons.

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